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ELEMENTS OF DYNAMICS

[TREATED ANALYTICALLY]

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PREFACE TO THE THIRD EDITION

The book has been thoroughly revised so as to eliminate the errors in the second edition. The subject matter has been re-arranged to make it more lucid. A number of new examples—both solved and unsolved have been added. It is hoped that the book will prove more beneficial to the students of Pre-Engineering and B.A./B.Sc., for whom it is intended.

We are extremely grateful to the teachers and students of Mathematics for their kind appreciation of the book.

5, June, 1971 *AUTHORS*

PREFACE

This text-book is an introduction to the study of Elements of Dynamics and is written entirely for the students preparing for Pre-Engineering and B.A./B.Sc. (Pass Course) Examination of Indian Universities. The subject matter is treated with Calculus wherever found necessary.

The text is illustrated by a large number of carefully solved examples. The exercises include a large number of questions set at the various university examinations.

We have freely used the standard works of modern writers to whom our acknowledgement is due.

Our thanks are due to the Publisher Sh. Ram Rang and the Printers who have taken great pains to bring this book in the present shape.

Any suggestion for improvement of the book will be gratefully received.

AUTHORS

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The following results may be used for numerical questions :

$\sqrt{2} = 1{\cdot}414$ $\pi = 3{\cdot}1416$

$\sqrt{3} = 1{\cdot}732$

$\sqrt{5} = 2{\cdot}236$ $\frac{1}{\pi} = 0{\cdot}31831$

$\sqrt{6} = 2{\cdot}449$

$\sqrt{7} = 2{\cdot}646$

$\sqrt{10} = 3{\cdot}162$ $\pi^2 = 9{\cdot}870$

CHAPTER I

INTRODUCTORY

VELOCITY AND ACCELERATION

(A) Preliminary Ideas

1·1 Dynamics is that branch of *mechanics* which treats of bodies in motion.

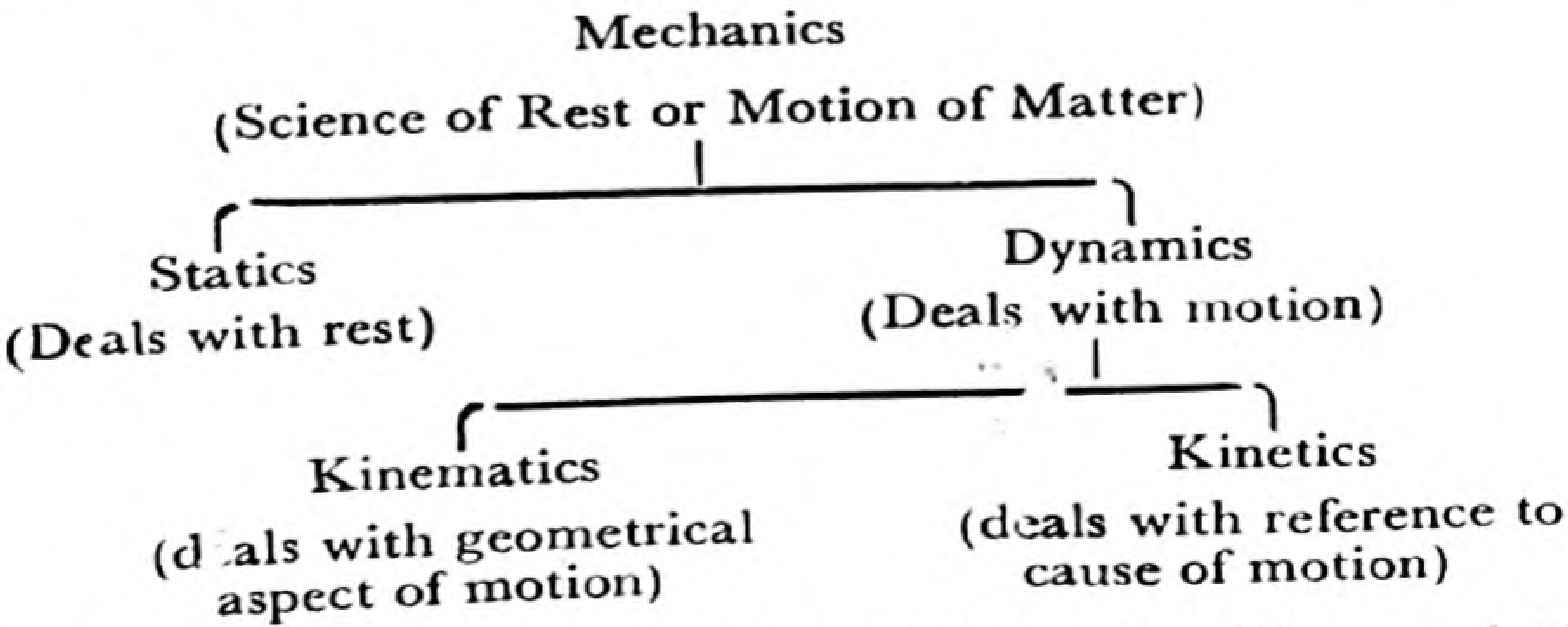
The branch which deals with equilibrium of bodies is known as Statics.

Two branches of Dynamics—Kinematics and Kinetics : The problem of motion has two aspects, *viz.*,

(*i*) Study of motion without any reference to force, the agent behind the motion. This geometrical aspect is dealt within **Kinematics**

(*ii*) Study of motion, taking into consideration the effects of force or forces which produce motion. This is dealt with in **Kinetics**.

1·11 The chart below will help to understand the subject-matter in a nut-shell :


Mechanics
(Science of Rest or Motion of Matter)
Statics
(Deals with rest)
Dynamics
(Deals with motion)
Kinematics
(deals with geometrical aspect of motion)
Kinetics
(deals with reference to cause of motion)

1·2 All Rest or Motion is Relative : When a body changes its position, with regard to its surrounding objects, we say it is in motion, otherwise at rest. There is no *Absolute Rest* in the Universe ; even the study table of our reader, fixed in a corner of the room, is revolving along with the earth around the sun and the sun itself around another star Vega and so on and so forth.

As a matter of fact, *the problem of rest or motion is the problem of action and mutual adjustment of forces.*

1·3 Force : The intuitive idea of "force" is present in the application of muscular strength to do any job, *e.g.* in moving one's hand, lifting a weight, paddling a cycle, stopping a rolling ball, striking a ball with a bat or hockey stick or breaking a piece of brick e c. ,etc.

1·31 Definition of Force : *Formally*, according to Newton, a Force is something which changes or tends to change the state of rest or of uniform motion of a body in a straight line.

If the forces acting on a body or particle balance one-another, the body remains at rest or in equilibrium. The branch which deals with set up of forces in equilibrium is called **Statics**, while that which treats of bodies in motion is known as **Dynamics**.

(B) Displacement, Velocity and Acceleration

1·4 Scalars and Vectors : Two types of quantities enter our mathematical discussions :

(*i*) those which have *magnitude* only—known as **Scalars**, *e.g.* mass, time, temperature, distance.

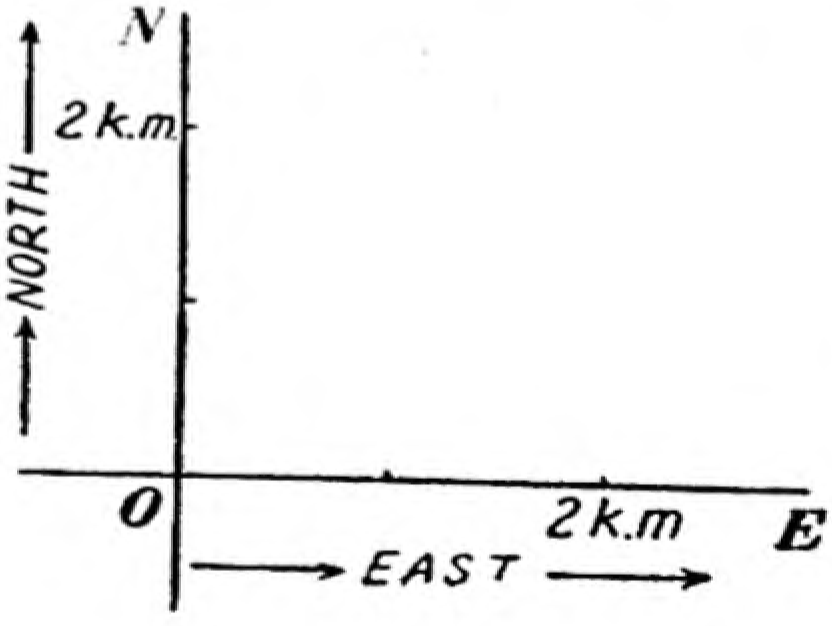
(*ii*) those which possess *direction* besides *magnitude*, *e.g.* displacement, velocity, acceleration, momentum, force, etc. They are called **Vectors**.

A vector, whose direction is from O to P and magnitude is the measure of OP, is symbolised as $\overrightarrow{\mathbf{OP}}$ or $\overline{\mathbf{OP}}$.

1·41 Distance and Displacement : If two persons start from O, one due north and the other due east and cover 2 kilometres, then their distances from O are same, but displecements are different.

Displacement is the distance covered in a particular direction.

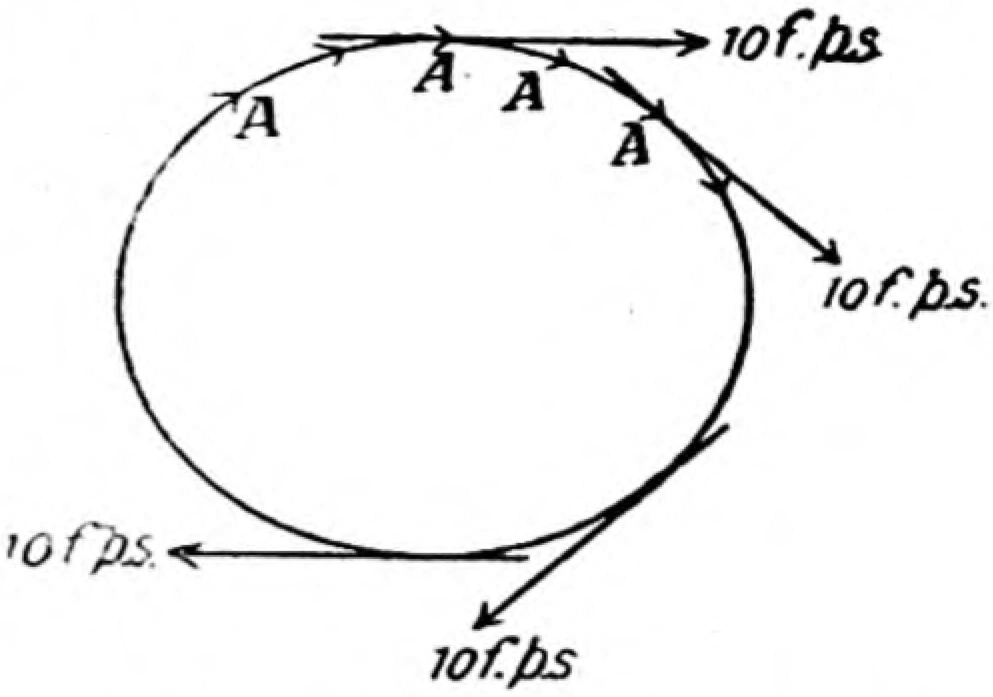
The line joining two adjacent points on a circle (or any other curve) is a tangent, hence the displacement at any instant of a particle moving along a circle (or a curve) is along the tangent.

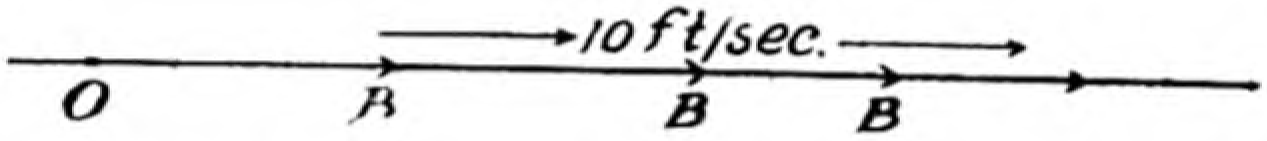


Distance is a Scalar while Displacement is a Vector quantity.

1·5 Speed and Velocity : The rate of change of *distance* is defined as **speed**, while the rate of change of *Displacement* is known as **Velocity.** If two persons, one moving east and the other south, move at the rate of 100 yards per minute, their *speeds are same* but *velocities are different.*

If a particle (A) moves along a circle at a constant rate of 10 ft. per second and another (B) along a st. line at the same rate ; then for A, *magnitude of velocity* (or speed) is same as for B ; but A changes its direction from point to point while B moves in the same direction. At any instant A is moving along the tangent at the point.





Speed is **Scalar** while velocity is a **Vector** quantity.

Speed is the name given to the magnitude of **Velocity.**

1·51 Constant or Uniform Speed : If a particle covers equal distances in equal intervals of time, however small, it is said to move with *uniform speed.*

If it moves in a constant direction also, it is said to have **uniform velocity.**

It is clear that if a particle moves in a st. line, its velocity and speed are same, while for motion in a curve speed means the magnitude of velocity ; the direction of velocity at any instant being along the tangent.

1·52 Average Velocity : The average velocity over an interval t, during which a body moves a distance S, is defined as

$$\frac{S}{t}, \ i.e. \ \frac{\text{Total displacement}}{\text{Time taken}}.$$

1·53 Units for Velocity or Speed : There are two types of units in use.

(*i*) In **F.P.S.** (*Foot-Pound-Second*) System, the unit used is one foot per second, written as 1 ft./sec or 1 f.p.s. and it sta the speed of a particle which moves one foot in one second.

(*ii*) In **C.G.S.** (*Centimetre-Gramme-Second*) System, the unit is one centimetre per second (1 cm./sec. or 1 c.p.s) which symbolises that a particle moves 1 cm in one second.

Thus 30 f.p.s. means that the particle or body moves 30 feet in one second. Similarly, 20 miles per hour or 20 m.p.h. means that

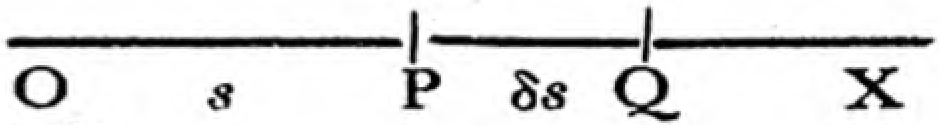
20 miles are covered in one hour. 12 ft./min. denotes a speed of 12 feet in one minute.

It is advantageous to remember that

60 m.p.h.=88 f.p s.
Thus 30 m.p.h.=44 f.p.s.
10 metre per sec.=36 km. per hour.

1·54 Analytical Expression for the Velocity of a Particle Moving in a Straight Line :

Let the particle move along the straight line OX, where O is the fixed point.



Let P (where OP=s) be the position of the point in time t and Q (where PQ=δs) its position at time $t+\delta t$.

Then displacement in time $\delta t=\delta s$.

∴ The rate of change of displacement or the average velocity during $\delta t=\frac{\delta s}{\delta t}$.

V, the velocity of particle at P is the limit of the average velocity when δt tends to zero.

$$\therefore \quad V=\text{velocity at } P=\underset{\delta t\to 0}{\text{Lt}}\frac{\delta s}{\delta t}$$

$$=\frac{ds}{dt}.$$

1·6 Acceleration : The rate of change of velocity is called *acceleration*. It may be either uniform or variable.

If a point moves so that the changes in velocity in equal intervals of time, however small, are equal amounts, then the *acceleration is said to be uniform*.

Retardation : If the sign of acceleration is opposite to that of velocity, *i.e.* it causes decrease in the magnitude of velocity, we call it **Retardation.**

In F.P.S. system, the acceleration is measured in feet per second per second which is usually written as ft /sec^2. or *f.p.s.s.* In C G.S. system it is measured in centimetres per second per second or cm./sec^2.

1·61 Analytical Expressions for Acceleration : Let a point move along a straight line OX, where O is a fixed point. Let at time t, P be the points of the particle and at time $t+\delta t$, let the particle be at Q.

Let OP=s and OQ=$s+\delta s$.

Let the velocities at P and Q be v and $v+\delta v$ respectively.

So that $$v=\frac{ds}{dt}.$$

Then the change in velocity in time $\delta t = \delta v$.

$\therefore$ the average acceleration or the average rate of change of velocity in time δt $= \dfrac{\delta v}{\delta t}$

$$\therefore f, \text{ the acceleration at } P = \underset{\delta t \to 0}{Lt} \frac{\delta v}{\delta t}$$

$$= \frac{dv}{dt}$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

Again $\quad f = \dfrac{dv}{dt} = \dfrac{dv}{ds} \times \dfrac{ds}{dt} = v\,\dfrac{dv}{ds}.$

$\therefore \quad f = \dfrac{dv}{dt}$ or $\dfrac{d^2s}{dt^2}$ or $v.\dfrac{dv}{ds}.$

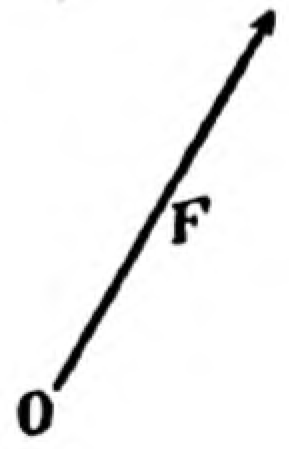
1·7 Representation of a Velocity by a St. Line : The velocity of a particle has (*i*) magnitude, (*ii*) direction, and (*iii*) point of origination (O).

A st. line drawn from O, in the direction of velocity and on a certain scale (*e.g.*, 10 ft./sec. being represented by 1 inch) will have a certain fixed length corresponding to the magnitude of the velocity and will thus completely represent the given velocity.

e.g., suppose a particle at O has a velocity of 54 f p.s. in the north-eastern direction.

Let us take a convenient scale, say, 18 f.p.s. may be represented by 1″. Then the line OA of length 3″ drawn from O in the north-eastern direction, represents the velocity of the particle.

Since all other *vectors*, acceleration, force, momentum etc., also possess magnitude and direction like velocity ; each one of them can also be represented completely by a segment of a straight line in the direction of the given vector, having a certain length, corresponding to the magnitude of the vector, on a *certain scale*.



Ex. 1. *If a body moves so that the distance s feet which it moves in t seconds is given by the relation $s = at^3$, find the velocity and the acceleration. Find the velocity and acceleration after 2 seconds.*

Sol. Here velocity, $V = \dfrac{ds}{dt} = 3at^2$

Velocity after 2 sec. $= 12a$ f.p.s.

and acceleration, $\quad f = \dfrac{d^2s}{dt^2} = 6at.$

Acceleration after 2 sec. $= 12a$ f.p.s.s.

Ex. 2. *If a particle moves along a straight line according to the law*

$$s^2=at^2+2bt+c,$$

prove that the acceleration varies as $\frac{1}{s^3}$.

Sol. Here $s^2=at^2+2bt+c.$

Differentiating both sides with respect to t

$$2s\frac{ds}{dt}=2at+2b,$$

or $$s.\frac{ds}{dt}=at+b.$$

Differentiating once again $w.r.$ to t

$$s.\frac{d^2s}{d^2t}+\left(\frac{ds}{dt}\right)^2=a$$

we get $$\frac{d^2s}{d^2t}=\frac{a-\left(\frac{ds}{dt}\right)^2}{s}$$

$$=\frac{a-\left(\frac{at+b}{s}\right)^2}{s}=\frac{as^2-(at+b)^2}{s^3}$$

$$=\frac{as^2-(a^2t^2+2abt+b^2)}{s^3}$$

$$\therefore \qquad f=\frac{ac-b^2}{s^3}\propto\frac{1}{s^3}.$$

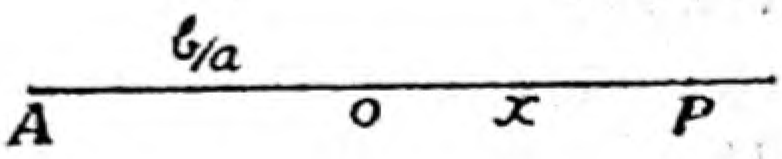
Ex. 3. *The velocity of a particle moving along a straight line is given by the relation*

$$v^2=ax^2+2bx+c$$

where x denotes the distance. Prove that the acceleration varies as the distance from a fixed point in the line (not the origin).

Sol. $v^2=ax^2+2bx+c.$ (Given)

Differentiating with respect to x



$$2v\frac{dv}{dx}=2ax+2b$$

or $$v\frac{dv}{dx}=ax+b$$

or $$f=a\left(x+\frac{b}{a}\right)$$

i.e. the acceleration varies as the distance from the point A, which is at a distance of $-\frac{b}{a}$ from O.

Ex. 4. *A particle moves along a straight line according to the law*

$$s=a\cos(\mu t+b)$$

show that the acceleration is directed to the origin and varies as the distance.

Sol. $$s=a\cos(\mu t+b) \qquad \dots(i)$$

Differentiating *w.r.t.* t, twice

$$\frac{ds}{dt}=-a\mu\sin(\mu t+b) \qquad \dots(ii)$$

$$\frac{d^2s}{dt^2}=-a\mu^2\cos(\mu t+b)$$

$$=-\mu^2 s.$$

Since velocity and acceleration are positive in the direction of s increasing, and s increases as the particle moves away from the origin, the negative sign on the right shows the acceleration is opposite to the direction for increase of s, that is, it is directed towards the origin.

Also, $\because$ $$\frac{d^2s}{dt^2}=-\mu^2\times\text{distance}$$

the acceleration varies as the distance.

Exercise I (A)

1 A particle moves in a straight line according to the law $s=\frac{1}{2}vt$ prove that its acceleration is constant.

2. A train travels half a mile in $93\frac{3}{4}$ seconds, find its speed in (*i*) miles per hour, (*ii*) kilometres per hour.

3. If a velocity of 24 kilometres per hour is the unit of velocity and kilometre, the unit of distance, how will you represent a velocity of 750 metres per minute.

4. The velocity of a body is 10 feet per sec. What is its measure in C.G.S. units ?

5. Two trains each 330 feet long are moving towards each other on parallel lines with velocities of 20 and 25 miles per hour respectively. Find the time that elapses from the instant when they first meet until they have cleared each other.

6. A particle moves in a straight line in such a way that the distance from a fixed point O in the line at the end of t seconds is given by $$s=7+3t+t^3.$$

Find (*i*) the distance from O,

(*ii*) its speed,

(*iii*) its acceleration,

at the end of 3 seconds.

7. A particle moves along a straight line according to the law $v^2=2(\sin x - x\cos x)$, where v is the velocity and x is the distance described. Find its acceleration.

8. A particle moves along a straight line, the law of motion being $s=a\sin(nt+k)$. Show that the acceleration is directed to the origin and varies as the distance.

9. Prove that a point cannot move so that its velocity varies as the distance from rest.

10. A point moves in a straight line so that its distance s from a fixed point at any time t is proportional to t^n. If v and f denote the velocity and acceleration at any time t, show that

$$(n-1)v^2=nsf.$$

11. The speed of a point in a line is given by the law

$$v^2=8x-2x^3.$$

Find its range of motion and show that

$$27v^4=4(4-f)(8+f)^2$$

where f denotes the acceleration

Answers

2. (*i*) 19·2 m.p.h., (*ii*) 30·897 km./hour.

3. By the number $1\frac{7}{8}$ or 1·875.

4. 50·75 cm./min. or 304·5 cm./sec.

5. 10 seconds.

6. $s=43$ ft. ; $v=30$ f.p.s. ; $f=18$ ft./sec^2.

7. $x\sin x$.

CHAPTER II

KINEMATICS OF MOTION IN A STRAIGHT LINE MOTION WITH CONSTANT ACCELERATION

2·1 Introductory Remarks : In majority of cases of motion in a straight line with which we come across in our every day life, the acceleration is constant or at least so for all theoretical purposes—*e.g.* for a body falling vertically or rolling along an inclined plane or even in cases of motion of cars and trains. Thus it would be advantageous to frame General Formulae to cover all such cases, where the acceleration is constant. We proceed to do so in the next Article.

(A) General Formulae

2·11 Motion in a Straight Line with Constant Acceleration : *A point moves in a straight line, starting with velocity* u, *and moving with constant acceleration* f *in its direction of motion ; if* v *be its velocity at the end of time* t, *and* s *be its distance at that instant from the starting point, then,*

(*1*) $v=u+ft$
(*2*) $s=ut+\frac{1}{2}ft^2$
(*3*) $v^2=u^2+2fs.$

Here $$\frac{d^2s}{dt^2}=f. \qquad \ldots(\text{A})$$

Integrating both sides, we get

$$\frac{ds}{dt}=ft+c,$$

where c is a constant to be determind from initial conditions

When $$t=0, \quad \frac{ds}{dt}=u \qquad (\text{Given})$$

$$\therefore \quad c=u$$

$$\therefore \quad \frac{ds}{dt}=u+ft$$

or $$\mathbf{v=u+ft} \qquad \ldots(i)$$

Integrating again,

$$s=ut+\tfrac{1}{2}ft^2+\text{A} \text{ ; A being a constant.}$$

When $$t=0,\ s=0. \qquad \therefore \quad t=0.$$

$$\therefore \quad \mathbf{s=ut+\tfrac{1}{2}ft^2} \qquad \ldots(ii)$$

Again, the acceleration can also be expressed as $v\dfrac{dv}{ds}$ so that the equation (A) can be written as :

$$v\frac{dv}{ds}=f. \quad \text{or} \quad v.dv=f.ds$$

Integrating, $\frac{1}{2}v^2=fs+B$, where B is a constant.

When $s=0, \quad v=0. \quad \therefore \quad B=\frac{1}{2}u^2.$

$\therefore \quad \frac{1}{2}v^2=fs+\frac{1}{2}u^2$

or $\quad \mathbf{v^2=u^2+2fs} \qquad \ldots(iii)$

This result could also be deduced from (*i*) and (*ii*) by eliminating t.

Thus

$$\begin{aligned} v^2 &= u^2+2uft+f^2t^2 \\ &= u^2+2f(ut+\tfrac{1}{2}ft^2) \\ &= u^2+2fs, \qquad \text{from } (ii) \end{aligned}$$

$\therefore \quad v^2=u^2+2fs.$

Cor. If the particle starts from rest, $u=0$, so that the above three equations become

$$v=ft$$
$$s=\tfrac{1}{2}ft^2$$

and $\quad v^2=2fs.$

Note. 1. These three formulae are the *fundamental formulae for motion* of a particle moving in a straight line with *constant acceleration.*

2. These equations are not to be applied in cases where the acceleration is variable.

3. Each of these equations contains four quantities so that any of the quantities can be found when the other three are given.

4. In working problems, care must be taken in selecting the equation which contains the given quantities and the quantity to be found.

Ex. 1. *A ball rolls down a hill from rest with uniform acceleration. Find the acceleration, if it travels 20 ft. in the first 5 seconds.*

Sol. The ball starts from rest

$\therefore \quad u=0$; also $s=2$ ft. ; $t=5$ sec.

We are required to find f.

Now $\quad s=ut+\frac{1}{2}ft^2$

Substituting the above values we obtain

$$20=0+\tfrac{1}{2}\times f\times 25$$

$\therefore \quad f=1{\cdot}6$ ft./sec^2.

Explanation : We are given the values of three of the four quantities (*viz.*, u, s, t) and require the value of the fourth, *viz.*, f. $\therefore$ we use the formula $s=ut+\frac{1}{2}ft^2$ which involves these four.

Ex. 2. *A cyclist riding at 7·5 miles per hour comes to the top of a slope of length 181·5 ft. If slope causes acceleration at the rate of 1 ft./sec², with what velocity will he reach the foot of the slope ?*

Sol. 7·5 miles an hour = 11 ft./sec.

Thus $u = 11$ ft./sec.

$s = 181{\cdot}5$ ft. and $f = 1$ ft./sec².

$\therefore$ $v^2 = u^2 + 2fs.$

Substituting the given values

$$v^2 = 121 + 2.1.\frac{363}{2}$$

$$= 121 + 363 = 484.$$

$\therefore$ $v = 22$ ft./sec.

i.e. 15 miles an hour.

Ex. 3. *The brakes are applied to a train travelling at 72 km. per hour, after passing over 200 metres its velocity is reduced to 36 km. per hour. At the same rate of retardation, how much further will it go before it is reduced to rest ?*

Sol. To determine the retardation produced by the brackes, consider the motion over 200 metres.

$u = 72$ km./hr. $= 20$ metres/sec.

$v = 36$ km./hr. $= 10$ metres/sec.

$s = 200$ metres.

$$v^2 = u^2 + 2fs$$

$$100 = 400 + 2f \times 200.$$

$\therefore$ $f = -\frac{3}{4}$ metre/sec².

Now $u = 36$ km./hr.

$= 10$ metres/sec.

$v = 0$

$f = -\frac{3}{4}$

$\therefore$ $v^2 = u^2 + 2fs.$

$$0 = 100 + 2(-\tfrac{3}{4})s.$$

$$s = 100 \times \tfrac{2}{3} = 66\tfrac{2}{3} \text{ metres.}$$

Ex. 4. *A point moves with uniform acceleration in a st. line and v_1, v_2, v_3 denote the average velocities in three successive intervals of time t_1, t_2, t_3. Prove that*

$$\frac{v_1 - v_2}{v_2 - v_3} = \frac{t_1 + t_2}{t_2 + t_3}.$$ *(Delhi, 1953 ; P.U.)*

Sol. Let the point be at A, at a certain instant. Its velocity is u and is moving with an acceleration f. It describes the distance s_1 in time t_1.

$$\therefore \quad s_1 = v_1 t_1$$

$$\therefore \quad v_1 t_1 = u t_1 + \tfrac{1}{2} f t_1^2.$$

Its velocity after time t_1 is $u+ft_1$. It describes a distance s_2 in time t_2

$$\therefore \quad s_2 = v_2 t_2$$

$$\therefore \quad v_2 t_2 = (u+ft_1)t_2 + \tfrac{1}{2} f t_2^2$$

Similarly,

$$v_3 t_3 = [u+f(t_1+t_2)]t_3 + \tfrac{1}{2} f t_3^2$$

or

$$v_1 = u + \tfrac{1}{2} f t_1 \qquad \ldots(1)$$

$$v_2 = u + f t_1 + \tfrac{1}{2} f t_2 \qquad \ldots(2)$$

$$v_3 = u + f t_1 + f t_2 + \tfrac{1}{2} f t_3 \qquad \ldots(3)$$

Subtracting (2) from (1) and (3) from (2), we get

$$v_1 - v_2 = -\tfrac{1}{2} ft(_1 + t_2) \qquad \ldots(4)$$

$$v_2 - v_3 = -\tfrac{1}{2} f(t_2 + t_3) \qquad \ldots(5)$$

Diuiding (4) by (5)

$$\frac{v_1 - v_2}{v_2 - v_3} = \frac{t_1 + t_2}{t_2 + t_3}.$$

Imp.

Ex. 5. *If the co-ordinates of a point moving with constant acceleration be x_1, x_2, x_3 at the instants t_1, t_2, t_3 respectively, prove that the acceleration is*

$$2.\frac{(x_2-x_3)t_1 + (x_3-x_1)t_2 + (x_1-x_2)t_3}{(t_2-t_3)(t_3-t_1)(t_1-t_2)}.$$

Sol. Let u be the initial velocity and f be the acceleration. Then, we have

$$x_1 = u + \tfrac{1}{2} f t_1^2$$

$$x_2 = u + \tfrac{1}{2} f t_2^2$$

$$x_3 = u + \tfrac{1}{2} f t_3^2$$

$$\therefore \quad x_2 - x_3 = u(t_2 - t_3) + \tfrac{1}{2} f(t_2^2 - t_3^2) \qquad \ldots(1)$$

$$x_3 - x_1 = u(t_3 - t_1) + \tfrac{1}{2} f(t_3^2 - t_1^2) \qquad \ldots(2)$$

$$x_1 - x_2 = u(t_1 - t_2) + \tfrac{1}{2} f(t_1^2 - t_2^2) \qquad \ldots(3)$$

Multiplying (1), (2), (3) by t_1, t_2, t_3 respectively and adding, we get

$$(x_2-x_3)t_1 + (x_3-x_1)t_2 + (x_1-x_2)t_3$$

$$= \tfrac{1}{2} f(t_2-t_3)(t_3-t_1)(t_1-t_2)$$

$$\therefore \quad f = 2.\frac{(x_2-x_3)t_1 + (x_3-x_1)t_2 + (x_1-x_2)t_3}{(t_2-t_3)(t_3-t_1)(t_1-t_2)}.$$

Ex. 6. *Two cars start off to race with velocities u and a and travel in a straight line with uniform accelerations α and β; if the race ends in a dead heat, prove that the length of the course is*

$$\frac{2(u-v)(u\beta - v\alpha)}{(\alpha-\beta)^2}.$$

Sol. Dead heat means that both the cars cover the same distance in the same time. Let s be the distance covered and t the time taken by both the cars. Then

$$s=ut+\tfrac{1}{2}\alpha t^2 \qquad \ldots(1)$$

Also $$s=vt+\tfrac{1}{2}\beta t^2 \qquad \ldots(2)$$

Subtracting,

$$0=(u-v)t+\tfrac{1}{2}(\alpha-\beta)t^2$$

which gives

$$t=\frac{2(u-v)}{\beta-\alpha}.$$

Using this value of t in (1) or (2), we get

$$s=\frac{2(u-v)(u\beta-v\alpha)}{(\alpha-\beta)^2}.$$

(B) Results for Motion with Constant Acceleration

2·2. To prove that the velocity at the middle of interval, mean of the intial and final velocities and the average velocity are equal to one another when a particle moves in a st. line with constant acceleration.

Let u be the initial velocity, t be the time interval and f the constant acceleration.

The final velocity v and the distance s travelled in t seconds are given by

$$v=u+ft \qquad \ldots(i)$$

$$s=ut+\tfrac{1}{2}ft^2 \qquad \ldots(ii)$$

v_1=velocity at the middle of the interval,

i.e. $\left(\frac{t}{2}\right)$ seconds will be

$$=u+f\left(\frac{t}{2}\right)$$

$$=u+\tfrac{1}{2}ft \qquad \ldots(iii)$$

v_2=mean of initial and final velocities

$$=\frac{u+v}{2}$$

$$=\tfrac{1}{2}(u+u+ft)=u+\tfrac{1}{2}ft \qquad \ldots(vi)$$

v_3=averge velocity

$$=\frac{\text{Total Displacement}}{\text{Total time}} \text{ or } \frac{s}{t}$$

$$=\frac{ut+\frac{1}{2}ft^2}{t}=u+\tfrac{1}{2}ft \qquad \ldots(v)$$

Thus we see that

$$v_1=v_2=v_3.$$

2·21 Distance travelled $=s=ut+\frac{1}{2}ft^2$

$$=(u+\tfrac{1}{2}ft)t$$

$$=\textbf{Mean velocity} \times \textbf{Time}$$

2·22 To find the distance traversed in the nth second of a body's motion.

(P.U. Pre-Engg. 1966)

Distance travelled in the nth second

= distance travelled in n second — the distance travelled in $(n-1)$ seconds

$$=un+\tfrac{1}{2}f(n^2)-\{u(n-1)+\tfrac{1}{2}f(n-1)^2\}$$

$$=u+\tfrac{1}{2}f(2n-1).$$

Velocity at the end of $(n-1)$ seconds

$$=u+f(n-1)$$

Velocity at the end of n seconds

$$=u+fn.$$

∴ average velocity during nth seconds.

$$=\tfrac{1}{2}[u+f(n-1)+u+fn]=u+\tfrac{1}{2}f(2n-1)$$

which is equal to the distance travelled in nth second.

2·23 To prove that distances travelled in successive seconds are in A.P.

(P.U. 1961)

S_1 = Distance travelled in nth sec. $=u+\left(\frac{2n-1}{2}\right)f.$

S_2 = ,, ,, ,, $(n+1)$th ,, $=u+\left(\frac{2n+1}{2}\right)f.$

S_3 = ,, ,, ,, $(n+2)$th ,, $=u+\left(\frac{2n+3}{2}\right)f.$

$$S_2-S_1=f.$$

$$S_3-S_2=f.$$

$$\therefore \quad S_2-S_1=S_3-S_2.$$

∴ S_1, S_2, S_3 form an A.P. with common difference f.

Exercise II (A)

1. A body starting from rest, moves with an acceleration equal to 30 cm./sec^2. Find the velocity at the end of 50 seconds and the distance described in that time.

2. The speed of a train increases from 36 km.p.h. to 54 km. per hour in 10 seconds. What is the acceleration of the train and what distance does it describe during that time ?

3. If a particle, moving with constant acceleration passes over 450 metres while its velocity increases from 25 to 35 metres per sec., find the acceleration and the time of motion.

✓. The velocity of a point moving in a st. line with constant retardation decreases 10 cm./sec. while it travels 1 metre and 15 cm./sec. while it travels 1·35 metres from the origin. Find retardation, initial velocity and the space it covers before coming to rest.

5. A train is moving with a speed of 52·5 m.p.h. and brakes produce a retardation of 3·5 ft /sec². At what distance from the station the brakes must be applied so that the train comes to rest at the station ?

6. A body moving with uniform acceleration and starting from rest describes 117 ft. in the 20th second. Find the acceleration.

7. An engine driver whose train is travelling at the rate of 30 m.p.h. sees a danger signal at a distance 220 yds. and does his best to stop the train. Supposing that he can stop the train when travelling 30 m.p.h. in 440 yards, show that his train will reach the signal with a velocity of $21\frac{1}{5}$ miles per hour nearly. *(P.U.)*

8. A train moving with constant acceleration takes t seconds to pass a signal while the ends of the train pass it with velocities u and v respectively. Find the (*i*) length of the train, (*ii*) its acceleration, (*iii*) velocity of the centre of the train while it passes the signal.

9. A particle moving with uniform acceleration in a straight line passes points P, Q, R. If PQ=QR=b, and if the time from P to Q be t_1 and from Q to R is t_2, prove that the acceleration is

$$\frac{2b(t_1-t_2)}{t_1t_2(t_1+t_2)}. \qquad \textit{(P.U. 1963)}$$

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10. If the distances gone over by the particle in the pth, qth and rth seconds respectively are x, y, z, show that

$$(q-r)x+(r-p)y+(p-q)z=0. \qquad \textit{(P.U.)}$$

11. A particle is moving in a straight line with constant acceleration. If it describes distances s and s' during the mth and nth seconds of its motion respectively, prove that its acceleration is

$$\frac{s-s'}{m-n}.$$

12. A body moving with uniform acceleration acquires a velocity of 10 ft./sec. after moving through 25 feet and a velocity of 12·5 feet/sec. after moving through a further distance of 25 feet. Find the acceleration and initial velocity.

13. A particle is projected in a straight line with a certain velocity and a constant acceleration. One second later another particle is projected with half the velocity and double the acceleration. When it overtakes the first, the velocities are 22 ft. and 31 ft. per second respectively. Prove that the distance described is 48 ft.

(D. U.)

14. A ball rolling down a slope with uniform acceleration passes a series of posts driven in the ground at equal distances. The velocities when passing three successive posts are v_1, v_2, v_3. Prove that

$$v_1^2+v_3^2=2v_2^2.$$

15. A body starts with velocity u and moves in a straight line with constant acceleration. When the velocity of the particle has increased to $4u$, the acceleration is reversed in direction. Find the velocity of the particle when it returns to the starting point.

(P.U. 1960)

16. A particle moving in a straight line with uniform acceleration f passes a certain point with velocity u. Three seconds later another particle, moving in the same st. line with acceleration $\frac{4}{3}f$, passes the same point with velocity $\frac{1}{3}u$. The first particle is overtaken by the second when their velocities are respectively 27 and 31 ft./sec. Find the values of u and f, and also the distance travelled from the point.

17. A cyclist travelling at a speed of 12 m.p.h. passes a car which is just starting with an acceleration of 1·6 f.p.s.s. The car maintains its acceleration for 20 seconds and then moves uniformly. How far will the car have run before overtaking the cyclist.

18. Two particles P, Q move along st. line AB, starting from A. P moves with velocity u and acceleration f; Q with a velocity u' and acceleration f'. If they both have the same velocity at the middle point of AB, prove that

$$AB=\frac{u'^2-u^2}{f-f'}.$$

(L.U.)

19. A 100 metre sprinter starts with a speed of 6 metre/sec. accelerating uniformly to 10 metre/sec. and finishes the race at this speed. If the total time taken in 12 sec., find the acceleration.

Answers

1. $v=15$ metre/sec. ; $s=375$ metre.

2. $f=\frac{1}{2}$ metre/sec.^{2} : $s=125$ metre.

3. $f=\frac{2}{3}$ metre/sec.^{2} ; $t=15$ sec.

4. $r=2\cdot5$ cm./sec.^{2} : $u=30$ cm./sec. ; and $s=180$ cm.

5. 847 feet. **6.** $f=6$ f.p.s.s.

12. $f=\frac{9}{8}$ ft./sec.^{2} : $u=\frac{5}{2}\sqrt{7}$ f.p.s.

15. $u\sqrt{31}$.

16. $u=3$ ft./sec. ; $f=\frac{1}{2}$ ft./sec.^{2} ; 720 ft.

17. $391\frac{1}{9}$ feet. **19.** $\frac{2}{5}$ meter/sec^2.

(C) Vertical Motion Under Gravity

2·3 Acceleration of falling bodies. It is known experimentally that whenever a body is let fall in vacuum, it will move vertically downward with an acceleration which is always the same at the same place of the earth. This acceleration, however, varies for different places and is independent of the mass of the body.

The value of this acceleration, called the *acceleration due to gravity,* in F.P.S. system is about 32·091 at the Equator, and in C.G.S. system it is 978.

The acceleration due to gravity is always denoted by g and in numerical problems its value is always taken as 32 ft./sec^2 in F.P.S. system and 981 cm./sec^2 in C.G.S. system.

2·31 Vertical motion under gravity. For a body projected vertically upward with velocity u, taking upward direction as positive direction, the value of acceleration is $-g$ so that the body moves with a constant retardation.

Thus the equations of motion become

$$h=ut-\tfrac{1}{2}gt^2 \qquad \ldots (i)$$

$$u=u-gt \qquad \ldots (ii)$$

and

$$v^2=u^2-2gh \qquad \ldots (iii)$$

Some important deductions are as follows :

1. *Time taken to reach the greatest height*

At the highest point, $v=0$

so that from (*ii*), $0=u-gt$

$$\therefore \quad t=\frac{u}{g}.$$

2. *Greatest Height*

Also from (*iii*), greatest height reached by the particle is given by

$$0=u^2-2gh$$

$$\therefore \quad h=\frac{u^2}{2g}.$$

3. *Time of flight* is the time taken by the particle to return to the point of projection.

Now putting $h=0$ in (*i*), we get

$$0=ut-\tfrac{1}{2}gt^2$$

either $t=0$ or $t=\dfrac{2u}{g}$,

$t=0$ refers to the instant when the body is projected and $t=\dfrac{2u}{g}$ is the time required by the body to return to the point of projection.

Thus $t=\dfrac{2u}{g}$ gives the time of flight.

Note that it is twice the time taken by the body to reach the greatest height.

4. *Time taken to attain a given height*: If h is the height attained in time t,

$$h=ut-\tfrac{1}{2}gt^2$$

so that $$gt^2-2ut+2h=0$$

which being a quadratic equation in t gives two values. The smaller value refers to the instant when the body attains the given height while moving upwards and the other value refers to the instant when the body is at the same height while moving downwards.

5. *Velocity at a given height.* From equation (*iii*),

$$v^2=u^2-2gh$$

$$\therefore \quad v=\pm\sqrt{u^2-2gh}.$$

The positive value of v is the velocity of the body when moving upward and the negative value of v is the velocity when the body is in downward motion. Thus in the downward direction the motion is exactly the reverse of the motion in the upward direction.

2·4 Motion of a body falling vertically downwards. Let the body be projected vertically downwards with velocity u. Then the positive direction is downwards in this case. The equations of motion are

$$h=ut+\tfrac{1}{2}gt^2$$

$$v=u+gt$$

$$v^2=u^2+2gh.$$

If the body falls freely from rest $u=0$.

Then the above equations become

$$h=\tfrac{1}{2}gt^2,\ \ v=gt \ \text{ and } v^2=2gh.$$

Now the velocity due to a fall from a height h is

$$v=\sqrt{2gh}$$

which is also the velocity required to take the body to the height h when projected vertically upwards.

Ex. 7. *If a stone is thrown vertically upwards with a velocity of 40 feet per second. Find the height to which it will rise and the whole time of flight.*

Sol. When the stone is at its greatest height its velocity is zero

$$0=40^2-2gh$$

$$\therefore \quad h=\frac{40\times 40}{2\times 32}.$$

$$\therefore \quad h=25 \text{ feet.}$$

Total time of flight is given by

$$0=ut-\tfrac{1}{2}gt^2 \text{ or } t=\frac{2u}{g}=\frac{80}{32}$$

Time of flight $=2\tfrac{1}{2}$ sec.

Ex. 8. *A body is projected vertically with velocity 40 ft. per sec. and 1 second later, another body is projected vertically upwards with a velocity of 80 ft. per sec., find when and where they will meet.*

Sol. Suppose the two bodies meet t seconds after the start of the second body. It means the first body has been in motion for $(t+1)$ seconds. The algebraic distance described by them is the same.

$$\therefore \quad 40(t+1)-\tfrac{1}{2}g(t+1)^2=80t-\tfrac{1}{2}gt^2$$

$$\text{or} \quad 40(t+1)-16(t^2+2t+1)=80t-16t^2$$

$$i.e. \quad 72t=24.$$

$$\therefore \quad t=\tfrac{1}{3} \text{ sec.}$$

The distance from the starting point to the position when the two bodies meet

$$=80\times\frac{1}{3}-16\times\frac{1}{9}=\frac{224}{9}=24\frac{8}{9} \text{ ft.}$$

Ex. 9. *A body projected vertically upwards from the top of a tower reaches the ground in t_1 seconds. If projected vertically downwards it reaches the ground in t_2 seconds. Prove that if simply let drop it would reach the ground in $\sqrt{t_1t_2}$ seconds.*

Sol. Let h be the height of the tower and u the velocity of projection then, we have

$$-h=ut_1-\tfrac{1}{2}gt_1^2 \quad \ldots(1)$$

$$h=ut_2+\tfrac{1}{2}gt^2 \quad \ldots(2)$$

$$\text{or} \quad \frac{h}{t_1}=-u+\tfrac{1}{2}gt_1$$

$$\frac{h}{t_2}=u+\tfrac{1}{2}gt_2.$$

$$\therefore \quad \frac{h}{t_1}+\frac{h}{t_2}=\tfrac{1}{2}g(t_1+t_2).$$

$$\therefore \quad h=\tfrac{1}{2}gt_1t_2$$

Hence if simply let drop it would reach the ground in $\sqrt{t_1t_2}$ seconds.

2·5 Relative motion of two falling bodies : Since the acceleration of gravity is the same for all bodies, the *relative acceleration of two bodies under gravity is zero.*

This principle is of great use in finding when and where two bodies projected in the same vertical line will meet, or in finding their distance apart at any given instant of time. Example 8 can be solved by this metod.

When the second body starts the first body is at a height of $40-\frac{1}{2}g=24$ feet and its velocity is $40-g=8$ ft./sec. Then the relative velocity of the second with respect to the first is $80-8=72$ ft./sec. and this remains constant as the relative acceleration is zero. Time taken by the second to describe 24 ft. $=1/3$ seconds. Hence the first body moves for 4/3 seconds before the two bodies meet.

Ex. 10. *A stone is dropped from the top of a tower 100 ft. high and at the same instant another stone is projected from the foot with a velocity of 80 ft. per sec, find when and where they meet.*

Sol. Initially the velocities of the stones are 0 and 80 ft./sec. Hence the lower one approaches the upper with relative velocity of 80 ft. per second. The original distance apart is 100 ft. Hence they will be together in $\frac{100}{80}$ sec, *i.e.*, in $1\frac{1}{4}$ sec.

In this time upper stone will have fallen through a distance

$$s=\tfrac{1}{2}\times 32\times(\tfrac{5}{4})^2=25 \text{ ft.}$$

Hence the stones meet 25 feet below the top and 75 ft. above the bottom of the tower.

Ex. 11. *A stone is dropped from a balloon at a height of 400 ft. above the ground and it reaches the ground in 6 seconds. Find the velocity with which the balloon was rising.*

Sol. Let the upward velocity of the balloon be u ft. per second. Then the stone starts with an upward velocity u and in 6 sec., it is at a distance of 400 below the point of projection.

Therefore, $s=ut+\frac{1}{2}$ ft.^{2} gives

$$-400=u\times 6-\tfrac{1}{2}\times 32\times 36$$

$$\therefore\quad 6u=576-400=176.$$

$$\therefore\quad u=29\tfrac{1}{3} \text{ ft. per sec.}$$

Note that minus sign is given to g and s in this problem because they are measured downwards and positive sign to u because it is measured upwards.

Exercise II (B)

1. A stone is thrown upwards with a velocity of 64 ft. per second. Find the greatest height and the time taken to reach it.

2. A ball is thrown vertically upwards with a velocity of 96 f.p.s. (*i*) How high will it rise? (*ii*) How long does it take to return to the point of projection? (*iii*) At what time will it be mid-way of its journey?

3. A body is projected upwards with a velocity of 128 ft. per sec. Find when it will be at the heights 220, 240, ft. above and 144 ft. below the point of projection respectively. Explain the negative result in the last case.

4. If a stone is thrown vertically upwards with velocity 80 ft. per second, and another is thrown up with the same velocity one second later, find when and where will they meet.

5. From the top of a tower 144 ft. high a body is dropped and at the same instant another is projected from the bottom and they meet half way up. Find the initial velocity of the projected body and its velocity when they meet.

6. A stone is let fall from the top of a tower 240 ft. high and another is projected with a velocity V from its foot. Find V if both meet at a height of 96 feet.

7. A stone dropped from the roof of a house falls past a window 8 ft. high in $\frac{1}{4}$ sec, find the height of the roof above the window.

8. A lift moves upward from the ground floor of a building with an acceleration of 4 ft. p s s After 10 sec. a ball is dropped from the lift. When will the ball strike the floor and when will the lift be at that instant ?

9. From a balloon ascending with a velocity of 32 ft. per second, a stone is let fall and reaches the ground in 17 seconds. How high was the balloon when the stone was dropped ?

10. A balloon has been ascending vertically at a uniform rate for $4\frac{1}{2}$ seconds and a stone let fall from it reaches the ground in 7 seconds. Find the velocity of the balloon and its height when stone was dropped. *(P.U. 1950)*

11. A falling particle in the last second of its fall passes through 224 feet. Find the height from which it fell and the time of its falling.

12. The distance travelled by a body in the 4th second of its fall is to the distance travelled in the last second but five as 1 : 3. Find the number of seconds for which the body travelled.

13. A stone is let fall from the top of Qutab Minar and during the last second of its flight it is observed to fall $\frac{104}{225}$ of the whole distance. Find the height of the Minar.

14. A particle is dropped from the top of a tower h ft. high and at the same moment another particle is projected upwards from the bottom. They meet when the upper one has described $\frac{1}{n}$ of the distance. Show that the velocities when they meet are in the ratio $2 : n-2$ and that the initial velocity of the lower is

$$\sqrt{\frac{ngh}{2}}. \qquad \textit{(P.U. 1957)}$$

15. A particle falls freely from a point A and a second particle is allowed to fall from B, at a distance of 5 feet vertically below A, at the instant when the first particle is half way between A and B. Find when and where they will be together and show that their velocities at that time will be as 3 : 1.

16. A stone is thrown vertically upward with such a velocity as will just take it to the level of the top of a tower 100 ft. high. Two seconds later another stone is thrown up from the same place with same velocity. Find when and where the stones will meet.

(P.U. Pre-Engg. 1966)

17. A particle is projected vertically upwards with a velocity of u ft./sec. and after t seconds another particle is projected from the same point with same velocity. Prove that they meet at a height of $\frac{4u^2-g^2t}{8g}$ ft. after a time of $\left(\frac{t}{2}+\frac{u}{g}\right)$ seconds.

18. A particle, projected vertically upwards, takes t seconds to reach a height h ft. It t' seconds is the time from this point to the ground again, prove that

$$h=\tfrac{1}{2}gtt'$$

and the greatest height $\qquad = \frac{g(t+t')^2}{8}.$ *(P.U., T.D.C. I)*

Show also that the velocity of the particle at a height $\frac{1}{2}h$ is

$$\tfrac{1}{2}g\sqrt{t^2+t'^2}$$

Answers

1. 64 ft. ; 2 sec. **2.** (*i*) 144 ft. (*ii*) 6 sec. (*iii*) $3\pm\frac{3}{\sqrt{2}}$ sec.

3. 2 5 sec. ; 3 sec. ; 9 sec. and -1 sec. which means that if the particle instead of having been projected downwards with the same velocity, it would have descended through 144 feet in one sec.

4. At a ht. of 96 ft. after $5\frac{1}{2}$ sec.

5. $48\sqrt{2}$ f.p.s. ; zero. **6** $V=80$ f.p.s. **7.** $12\frac{1}{4}$ ft.

8. After 5 sec. The lift will be at a height of 450 feet.

9. 408 feet. **10.** $68\frac{4}{23}$ f.p.s. ; $306\frac{18}{23}$ ft.

11. 900 ft. ; $7\frac{1}{2}$ sec. **12.** 16 sec. **13.** 225 feet.

15. $\frac{1}{8}$S below B ; after $\frac{1}{16}\sqrt{2S}$ sec. of the start of second.

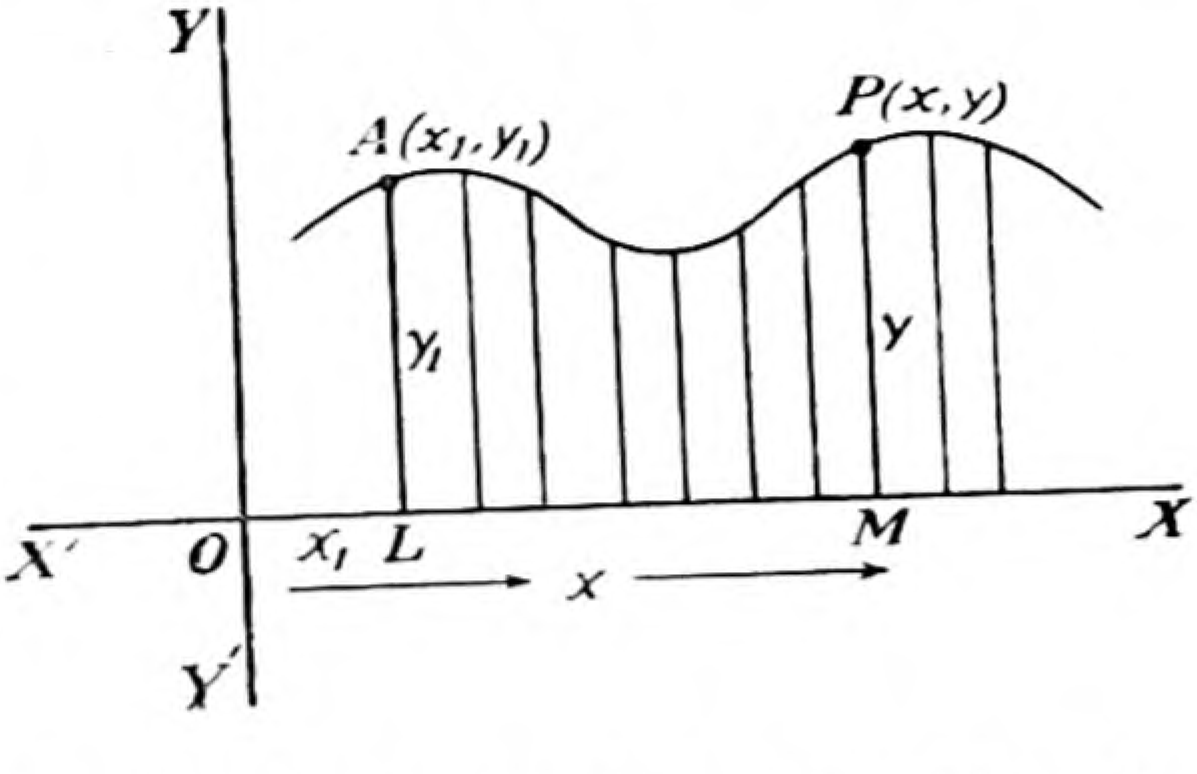
16. 84 ft. above the point of projection and $3\frac{1}{2}$ sec. after the instant of projection of the first stone.

(D) Graphical Kinematics

2·6 Graph of a function $y=f(x)$. If we give different values to x, and get the corresponding values of y from the functional relation

$$y=f(x) \qquad \ldots(i)$$

We shall get pairs of values (x_1, y_1), (x_2, y_2), (x_3, y_3).........(x_n, y_n) each pair giving rise to a point in the plane. By plotting these points w.r.t. a rectangular set of axes XOX′, YOY′ in the plane, and joining them by means of a free hand curve, we get what is called the graph of $y=f(x)$.



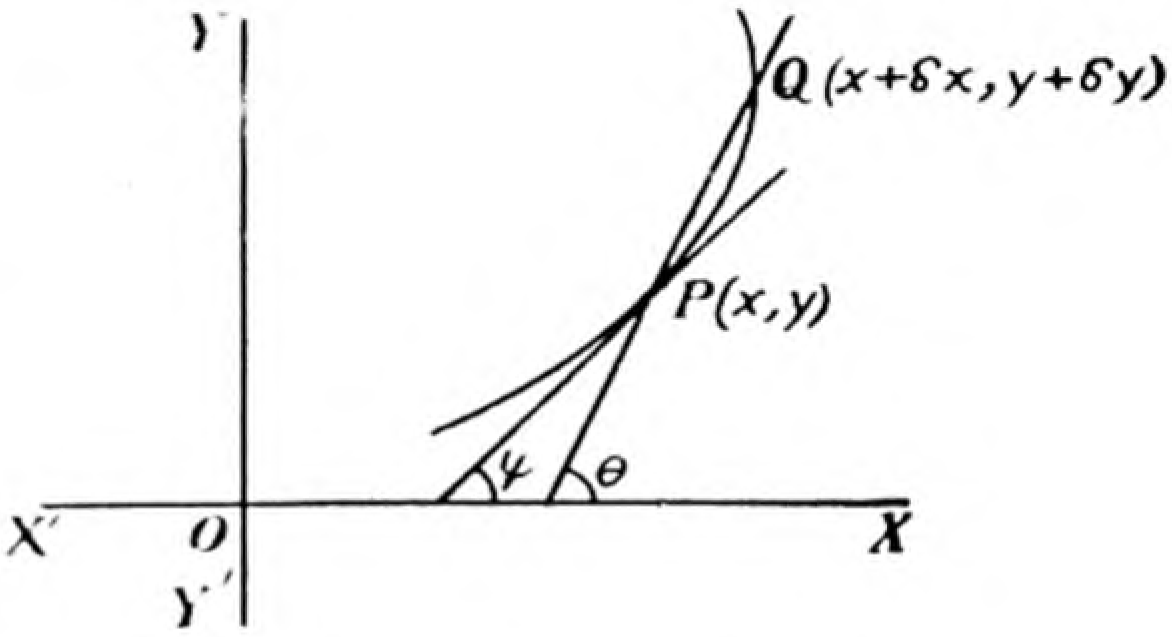
2·61 Geometrical Meaning of Derivation. Consider the continuous function

$$y=f(x) \qquad \ldots(i)$$

Take a point P(x, y) on its graph.

Let δx be a small change in the value of x and δy the corresponding small change (because y is a continuous function of x) in the value of y.

Then Q$(x+\delta x, y+\delta y)$ will be a point on the graph contiguous to P(x, y).



Let the chord PQ make an angle θ with OX, so that

$$\tan\theta=\frac{y+\delta y-y}{x+\delta x-x}=\frac{\delta y}{\delta x}.$$

As $\delta x\to 0$ and consequently, because y is a continuous function of x, $\delta y\to 0$; the point Q moving along the curve gradually coincides with P and QP in the limiting position becomes the tangent at P, making an angle ψ with OX, so that

$$\underset{\delta\to 0}{\text{Lt}}\ \frac{\delta y}{\delta x}=\underset{\theta\to\psi}{\text{Lt}}\ \tan\theta$$

or

$$\frac{dy}{dx}=\tan\psi.$$

Thus the derivative $\frac{dy}{dx}$ or $f'(x)$ *represents the slope of tangent at a point* (x, y) *on the graph of function* $y=f(x)$.

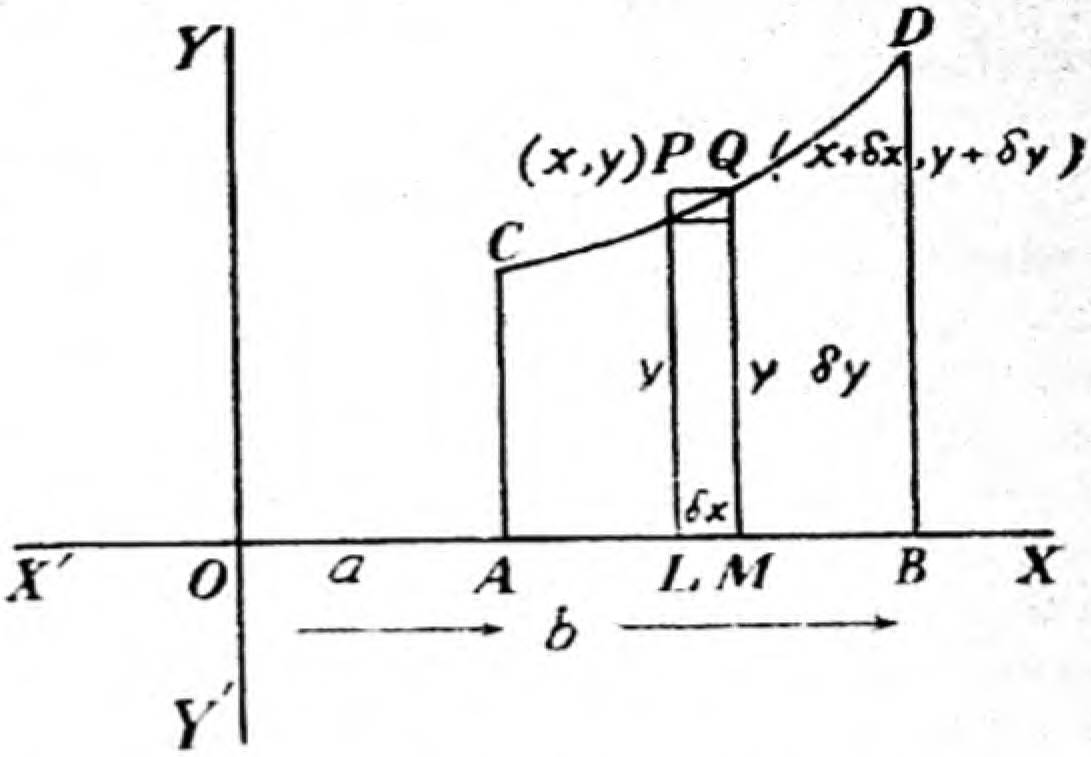
2·62 Geometrical Meaning of Integral : $\int_a^b y\,dx$:

Draw the graph of the function

$$y=f(x) \qquad \ldots(i)$$

Erect the ordinates AC and BD at the points A and B, where OA$=a$, OB$=b$.

Take points P(x, y) and Q$(x+\delta x, y+\delta y)$ on the graph and drop perpendiculars PL, QM on X-axis. Obviously, the area ALPC depends on the position of P and hence it is a function of x.



Let area ALPC be denoted by S and δS be the corresponding increase in the area by shifting from point P to Q, so that

$$\text{Area AMQC}=S+\delta S$$

or
$$\text{Area LMQP}=\delta S \qquad \ldots(i)$$

From the figure δS lies between two rectangles whose areas are $y.\delta x$ and $(y+\delta y)\delta x$.

$$y\,\delta x<\delta S<(y+\delta y)\delta x$$

or
$$y<\frac{\delta S}{\delta x}<y+\delta y.$$

In the limiting position, as Q→P, so that $\delta x\to 0$, $\delta y\to 0$;

$$\underset{\delta x\to 0}{\text{Lt}}\ \frac{\delta S}{\delta x}=y$$

i.e.
$$\frac{dS}{dx}=y$$

or
$$S=\int_a^x y\,dx+k.$$

To find k, we observe that $x=a$, $S=0$, so that $k=0$.

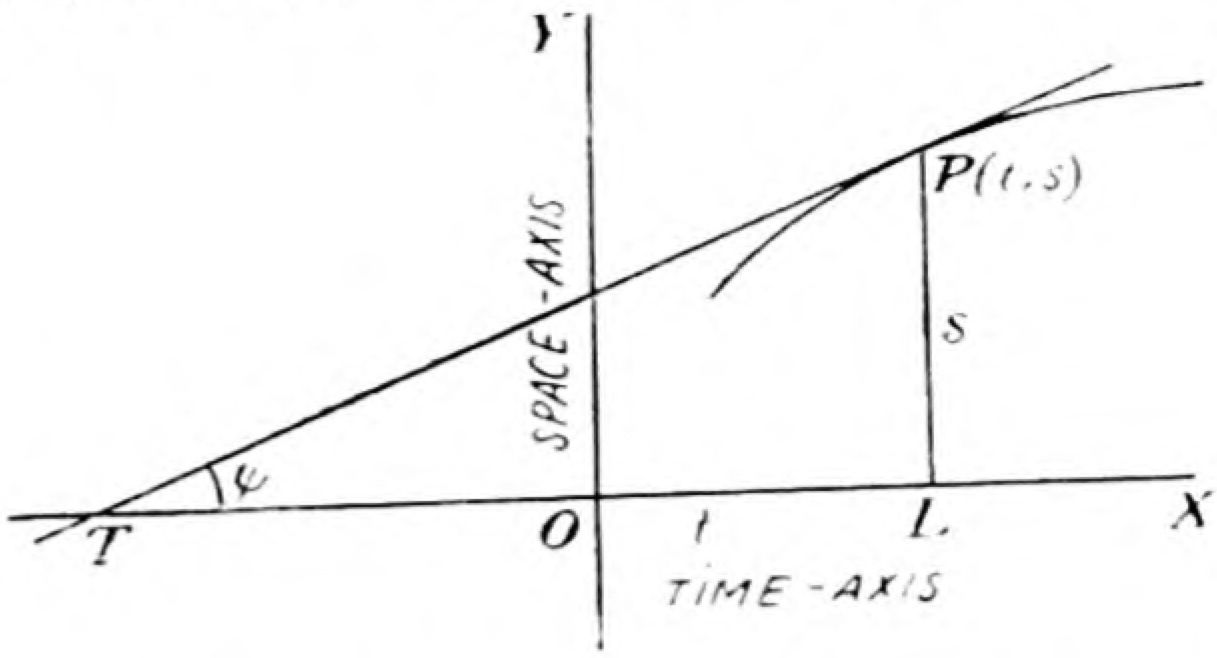
$$\text{Area ALPC}=S=\int_a^x y\,dx.$$

Hence
$$\text{Area ABDC}=\int_a^b y\,dx.$$

2·7 Space Time Graph : Let the space-time relation for the motion of a particle be

$$s=f(t) \qquad \ldots(i)$$

By giving different values to t, such as $t_1, t_2, t_3, \ldots\ldots$, we obtain the corresponding values $s_1, s_2, s_3, \ldots\ldots$, of s giving rise to pairs of co-ordinates (t_1, s_1), (t_2, s_2), $(t_3, {}_3)s$, ..., each pair giving rise to a point in the plane, with reference to a pair of Rectangular Axes XOX′, YOY′ (known as Time Axis and Space Axis respectively).



The free hand curve obtained by joining these periods is called *Space-Time Graph* for the motion of the particle.

The abscissa OL $(=t)$ of a point P, represents the **time** while the ordinate LP $(=s)$ exhibits the **distance** covered in time t.

How is Velocity represented in Space-Time Graph ?

$$\text{Velocity at time } t=\frac{ds}{dt}$$

$$=\text{Slope of the tangent at } P(t, s)$$

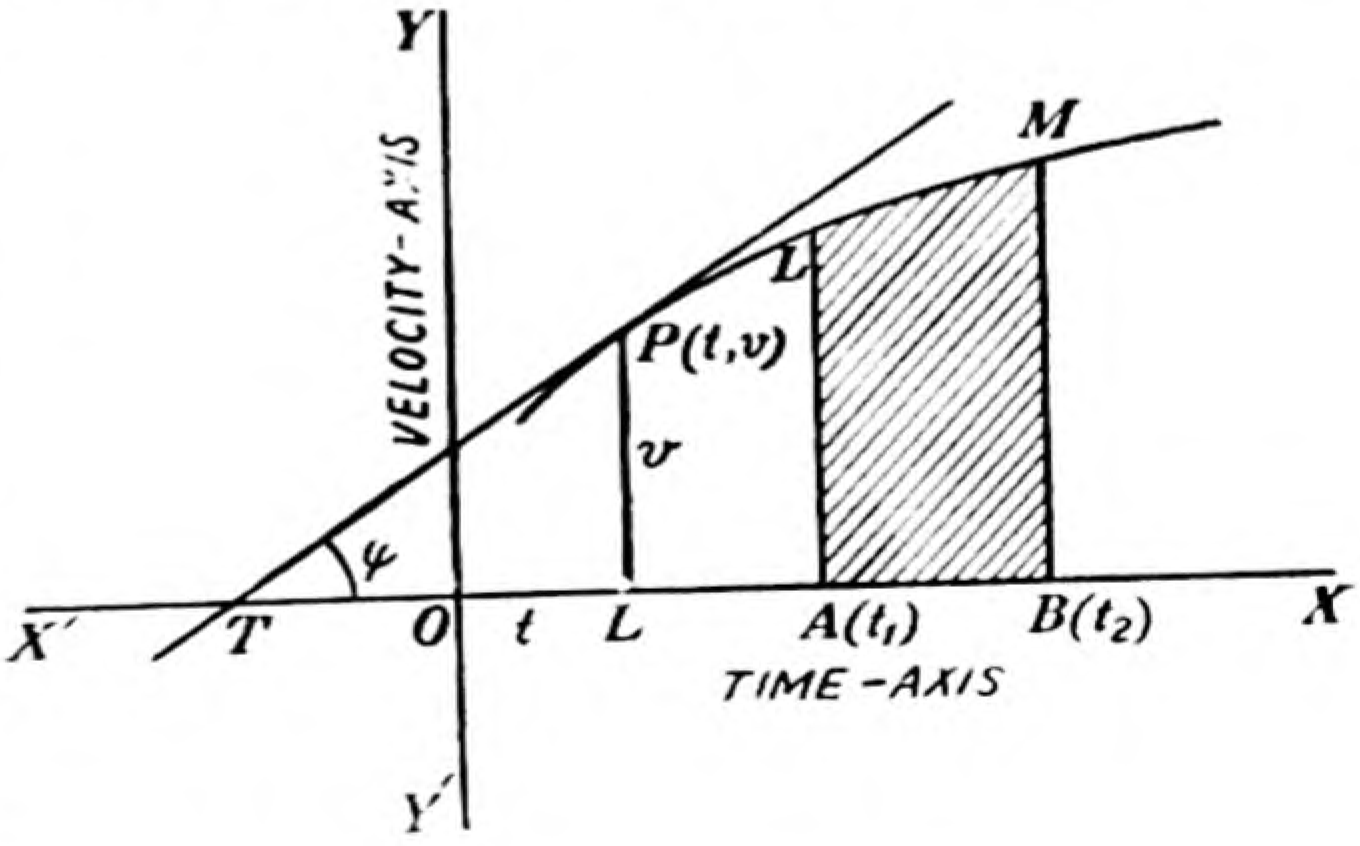
$$=\tan\psi.$$

Thus the velocity at an instant t is represented by the slope of the tangent to the Space-Time Graph at the point oorresponding to t.

2·8 Velocity-Time Graph : If the velocity-time relation for the motion of a particle be

$$v=f(t) \qquad \ldots(i)$$

we can draw the velocity-time graph in a similar manner by giving different values to t and obtaining the corresponding values of v ; each pair giving rise to a point in the plane, *w.r.t.* a set of perpendicular axes (Time Axis and Velocity Axis).



By plotting these points in the plane and joining them by a free hand curve we get the *Velocity-Time Graph*.

The *abscissa* OL $(=t)$ of a point P, represents the time and *ordinate* LP $(=v)$ represents the velocity at the instant t.

(*a*) **How is Acceleration represented in Velocity-Time Graph ?**

$$f\text{ (acceleration)} = \frac{dv}{dt}$$
$$= \text{Slope of the tangent at P}$$
$$= \tan \psi.$$

Thus acceleration at an instant t is represented by the slope of the tangent to the Velocity-Time Graph at the point P, corresponding to t.

(*b*) **How is Space (Distance travelled) represented ?**

Since $$v = \frac{ds}{dt}.$$

$\therefore$ $$s = \int v\, dt.$$

Suppose we wish to find the distance travelled from the instant t_1 to t_2.

$$S = \int_{t_1}^{t_2} v\, dt$$

= Area under the velocity-time graph surrounded by ordinates $t = t_1$ and $t = t_2$

= Area ABML.

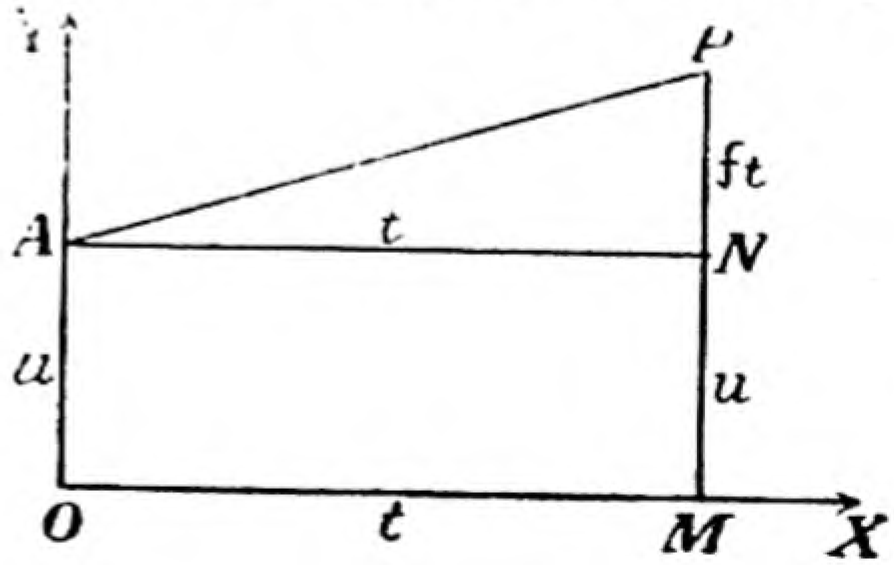
2·81 Velocity-Time Graph for the motion of a particle whose acceleration is constant : Since acceleration is represented by the slope of the tangent to the velocity-time graph, the graph should be such that it has uniform slope, *i.e.* a straight line.

Further if f is the acceleration, then the slope of the line must be f.

Thus, the velocity-time graph for a constant acceleration (f) is the straight line having its slope equal to f.

2·82 Graphical Proof of General Formulae for Motion with a Constant acceleration f : When the particle is moving with a constant acceleration f with initial velocity u, the slope of the velocity-time graph is the same everywhese. The graph is, therefore, a straight line with slope f.

Let OA $= u$ represent the initial velocity and MP $= v$ represent the final velocity. OM $= t$ represents the time.



Then $$\tan \angle NAP = f = \frac{NP}{AN}$$

$\therefore$ $$NP = f.AN = ft$$

Also $$NP = MP - MN$$
$$= v - u$$

$$\therefore \qquad v-u=ft$$

or

$$v=u+ft \qquad \ldots(i)$$

Distance described

$$\begin{aligned}
&=\text{area OMPA}\\
&=\text{area OMNA}+\text{area of }\triangle\text{ANP}\\
&=\text{OA}\cdot\text{OM}+\tfrac{1}{2}\text{AN}\cdot\text{NP}\\
&=ut+\tfrac{1}{2}t\cdot ft\\
&=ut+\tfrac{1}{2}ft^2
\end{aligned}$$

$$\therefore \qquad s=ut+\tfrac{1}{2}ft^2 \qquad \ldots(ii)$$

Area of trapezium OMPA

$$\begin{aligned}
&=\tfrac{1}{2}(\text{OA}+\text{MP})\text{OM}\\
&=\tfrac{1}{2}(\text{OA}+\text{MP})\text{AN}\\
&=\tfrac{1}{2}(\text{OA}+\text{MP})\text{NP}.\frac{\text{AN}}{\text{NP}}\\
&=\tfrac{1}{2}(\text{OA}+\text{MP})(\text{MP}-\text{MN})\cdot\frac{\text{AN}}{\text{NP}}\\
&=\tfrac{1}{2}(u+v)(v-u)\,\frac{1}{f}
\end{aligned}$$

$$s=\tfrac{1}{2}(u^2-v^2)\cdot\frac{1}{f}$$

or

$$v^2-u^2=2fs \qquad \ldots(iii)$$

2·9 To compare the space-average of the velocity over a distance, with the time-average where U and V are initial and final velocities of a point moving in a straight line with uniform acceleration.

Let v be the velocity at an instant t when the particle has travelled a distance s from the starting point.

If total distance travelled $=d$ and total time T

$$\text{Space-average of velocity}=\frac{1}{d}\int_0^d v\,ds \qquad \ldots(i)$$

$$\text{Time-average of velocity}=\frac{1}{\text{T}}\int_0^{\text{T}} v\,dt \qquad \ldots(ii)$$

If f is the constant acceleration, then

$$v=\text{U}+ft \quad \text{so that} \quad \text{V}=\text{U}+f\text{T} \qquad \ldots(iii)$$

$$v^2=\text{U}^2+2fs \text{ so that } \text{V}^2=\text{U}^2+2fd \qquad \ldots(iv)$$

$$\begin{aligned}
\therefore \qquad \text{Space-average}&=\frac{1}{d}\int_0^d \sqrt{\text{U}^2+2fs}\,ds\\
&=\frac{1}{3fd}\Big[\big(\text{U}^2+2fs\big)^{\frac{3}{2}}\Big]_0^d\\
&=\frac{1}{3fd}\Big[\big(\text{U}^2+2fd\big)^{\frac{3}{2}}-\text{U}^3\Big]\\
&=\frac{2}{3}\left[\frac{\text{V}^3-\text{U}^3}{\text{V}^2-\text{U}^2}\right]\\
&=\frac{2}{3}\cdot\frac{\text{U}^2+\text{UV}+\text{V}^2}{\text{U}+\text{V}}. \qquad \ldots(v)
\end{aligned}$$

$$\text{Time-average} \quad =\frac{1}{T}\int_0^T (U+ft)dt=\frac{1}{2fT}\Big[(U+ft)^2\Big]_0^T$$

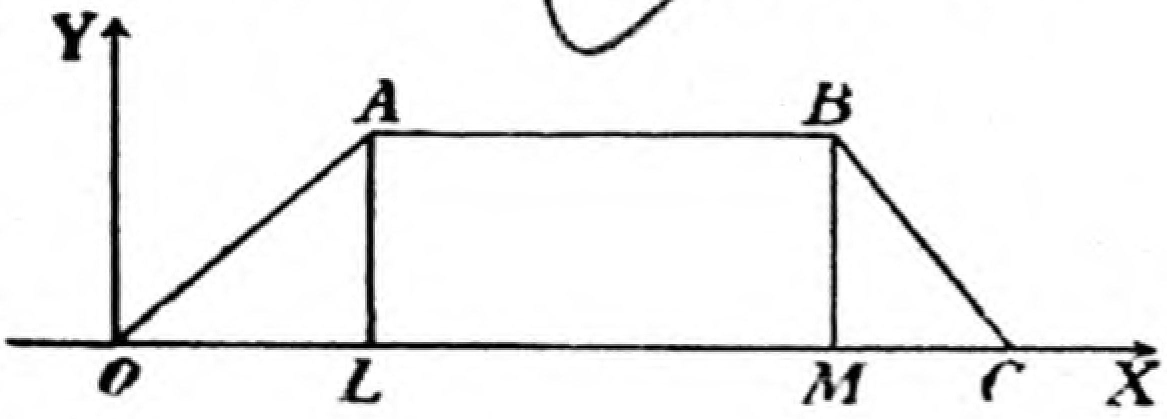
$$=\frac{1}{2fT}\Big[(U+fT)^2-U^2\Big]$$

$$=\frac{1}{2}\Big[\frac{V^2-U^2}{V-U}\Big]=\frac{V+U}{2} \qquad \ldots(vi)$$

Ex. 12. *A train starts from one station to reach another at distance c, the motion is first uniformly accelerated for time t, the velocity then remains constant for time t′ and is then retarded uniformly for a time t″.*

Find graphically the values of the acceleration and retardation.

Sol. Let OABC represent the velocity-time graph. AB is horizontal.



Draw AL, BM perpendicular to OX.

c = area OABC = $\triangle$OLA + rectangle ALMB + $\triangle$BMC

$$=\tfrac{1}{2}\text{OL.AL}+\text{AL.LM}+\tfrac{1}{2}\text{MC.BM}$$

$$=\left\{\frac{t+t''}{2}+t'\right\}\text{AL}.$$

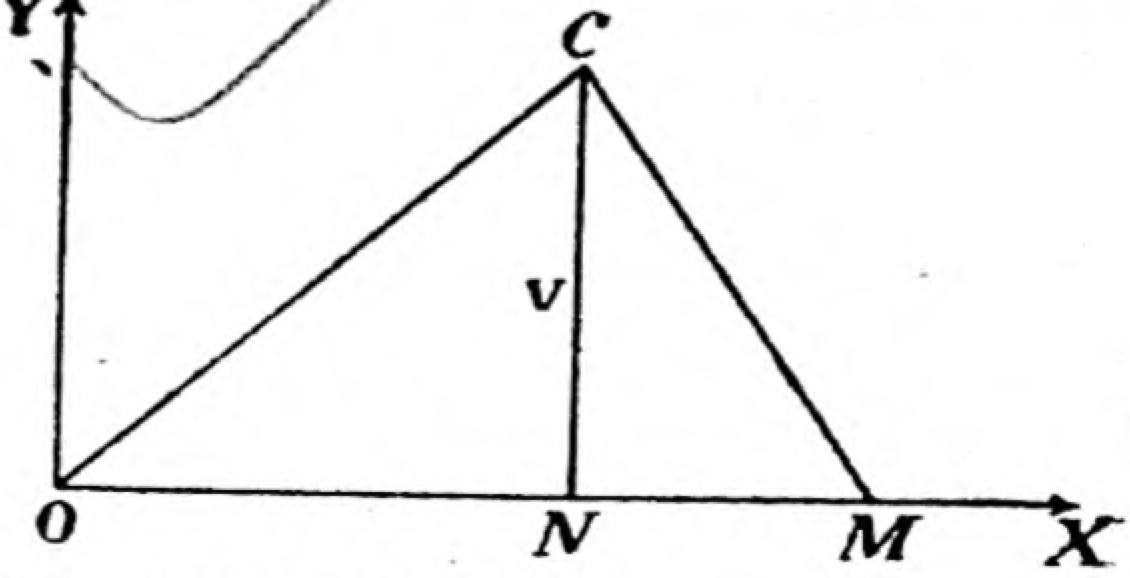
The acceleration during t is the slope of OA

$$=\frac{\text{AL}}{\text{OL}}=\frac{\text{AL}}{t}+\frac{2c}{(t+t''+2t')t}.$$

Retardation during t'' is the slope of BC

$$=\frac{\text{BM}}{t''}=\frac{2c}{(t+2t'+t'')t''}.$$

Ex. 13. *A train travels a distance S starting from rest and finishing at rest. The velocity increases uniformly till it reaches a maximum velocity V, and then decreases uniformly. Show that the whole time of the motion is* $\frac{2S}{V}$



Sol. The velocity-time graph is OCM. NC represents the maximum velocity V.

OM = the whole time

$$\text{S}=\text{Area OCM}=\tfrac{1}{2}\text{OM}\times\text{NC}$$

$$\text{S}=\tfrac{1}{2}\text{V}.t. \qquad \therefore \qquad t=\frac{2\text{S}}{\text{V}}.$$

Exercise II (C)

1. A bus travels a distance of 5·92 km. between two stops in 10 minutes. It constantly accelerates for 10 seconds to a maximum speed at which it runs for some time. Then it is brought to rest in 6 sec. under a constant retardation. Find the acceleration, retardation, and constant speed.

2. A train travels in 7 minutes and 45 seconds a distance of 5 miles from rest at one station to rest at another station. It is uniformly accelerated for the first 30 seconds and retarded for the last 20 seconds, the speed being constant for the remaining time. Find the acceleration, the retardation and the maximum velocity.

3. A train starts with uniform acceleration. then moves with a uniform velocity for 10 minutes ~~some time~~ and finally comes to rest under uniform retardation. If the total journey is 9 miles and time taken is 14 minutes ; show that the maximum velocity attained is 45 m p.h.

4. A lift performs the first part of its ascent with uniform acceleration f and the remainder with uniform retardation $2f$. Prove that if h is the depth of the shaft and t the time of ascent, $h=\frac{1}{3}ft^2$.

5. A train passes a station A at 30 miles an hour and maintains this speed for 7 miles and then is uniformly retarded stopping at B which is 8 miles from A. A second train starts from A at the instant the first passes and being uniformly accelerated for part of the journey and uniformly retarded for the rest, stops at B at the same time as the first. Find its greatest speed.

6. The greatest permissible acceleration and retardation of a train are 2 f.s.s. and 4 f.s.s. respectively and the maximum speed it can develop is 30 m.p.h. Find the least time taken to run between Delhi and Gaziabad, 12 miles apart.

7. A lift ascending from a pit 640 feet deep rises during the first part of the ascent with uniform acceleration. On nearing the top the upward force in cut off and the impetus of the lift is just sufficient to carry it to the top. If the whole process occupies one minute find the maximum velocity attained and acceleration during the first part of ascent. (*P.U. Pre. Engg. 1967*)

8. A train starts from rest and moves with a constant acceleration f for some time and then comes to rest after moving for the rest of time with a retardation r. If the total distance travelled is s and t is the time taken, prove that

$$t=\sqrt{2s\left(\frac{1}{f}+\frac{1}{r}\right)}\,.$$

by yourself (*P.U. 1961*)

9. A cage moving vertically ascends with a constant acceleration f, then with a uniform velocity and finally stops under a constant

retardation f. If the total height h is ascended in time t, show that the time during which the lift ascended with constant velocity is

$$\left(t^2-\frac{4h}{f}\right)^{\frac{1}{2}}. \qquad \text{(P.U.)}$$

10. Two stopping points of an electric tram car are 440 yards apart. The maximum speed of the car is 20 m.p.h. and covers the distance between stops in 75 seconds. If both acceleration and retardation are uniform and the latter is twice as great as the former, find the value of each of them and also calculate how far the car runs at its maximum speed. *(P.U., D.U.)*

11. For $\frac{1}{p}$ of the distance between two stations, a train is uniformly accelerated and for $\frac{1}{q}$ of the distance, it is uniformly retarded. It starts from rest at one station and comes to rest at another.

Prove that the ratio of its greatest velocity to the average velocity is

$$1+\frac{1}{p}+\frac{1}{q} : 1. \qquad \text{(P.U.)}$$

12. The speed of a train increases at a constant rate α from 0 to v, then remains constant for an interval and finally decreases to zero at a constant rate β. If l be the total distance described, prove that the total time occupied is

$$\frac{l}{v}+\frac{1}{2}\,v\left(\frac{1}{\alpha}+\frac{1}{\beta}\right). \qquad \text{(P.U.)}$$

Answers

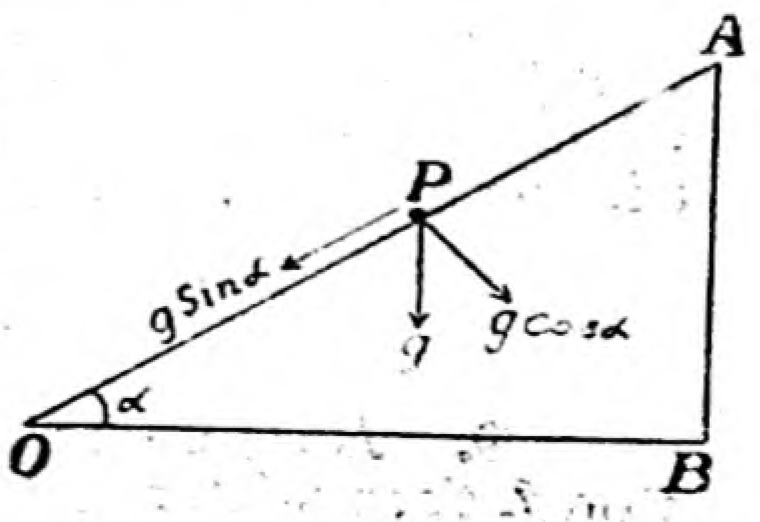
1. $f=1$ metre/sec². ; $r=\frac{5}{3}$ metres/sec². ; $V=36$ km.p h.
2. $f=2$ ft./sec². ; $r=3$ ft./sec². ; $V=40\frac{10}{11}$ m.p.h.
5. $53\frac{1}{3}$ m.p.h.
6. 24 min. $16\frac{1}{2}$ sec.
7. 2[illegible]$\frac{1}{3}$ ft./sec. ; $\frac{32}{89}$ ft /sec².
10. $f=\frac{11}{15}$ f.p.s.s. ; $r=\frac{22}{15}$ f.p.s. ; 440 feet.

(E) Motion down a Smooth Inclined Plane

2·10 Let α be the inclination of the plane with the horizontal and P a particle on the plane. The vertical acceleration due to gravity, g, of P can be resolved into two components,

(*i*) an acceleration $g \cos \alpha$ perpendicular to the plane.

(*ii*) an acceleration $g \sin \alpha$ down the plane.



Now the motion perpendicular to the plane is prevented by the plane itself. Thus the particle will move down the plane with an acceleration $g \sin \alpha$.

If l be the length of the plane OA and the particle starts from A with velocity u, its equations of motion are

$$x=ut+\tfrac{1}{2}g^{\prime} \sin^2 \alpha$$
$$v=u+gt \sin \alpha$$
$$v^2=u^2+2g\,x \sin \alpha.$$

If the particle starts from rest from A, then its velocsty on reaching O, the foot of the inclined plane, is given by

$$v^2=2g \sin \alpha\, l$$
$$=2gh$$

where AB$=h$; AOB being the right-angled triangle.

$$\therefore \qquad v=\sqrt{2gh}.$$

Thus the velocity at O is the same as the velocity acquired in falling freely through a height equal to the height of the plane. But the time taken to slide down AO is different from that taken to fall freely through the height AB. For the time taken to slide down AO is given by the equation

$$x=ut+\tfrac{1}{2}gt^2 \sin \alpha$$

where $u=0$ and $x=l$

$$\therefore \qquad l=\tfrac{1}{2}g \sin \alpha . t^2$$

or

$$t=\frac{2l}{\sqrt{g \sin \alpha}}$$

The time taken to fall freely a height h is given by

$$h=\tfrac{1}{2}gt^2.$$

$$\therefore \qquad t=\sqrt{\frac{2h}{g}}=\sqrt{\frac{2l \sin \alpha}{g}}.$$

If the particle is projected up the plane with velocity u, it will move with constant retardation $g \sin \alpha$ so that the equations of motion now are

$$x=ut-\tfrac{1}{2}gt^2 \sin \alpha$$
$$v=u-gt \sin \alpha$$
$$v^2=u^2-2gx \sin \alpha.$$

Ex. 14. *A body moves up a smooth inclined plane starting with a velocity of 48 ft. per sec. If the plane is inclined at an angle of 30°, find the velocity of the body when it has moved a distancs of 64 ft. from the starting point.*

Sol. For motion up the plane, we have

$$v^2=u^2-2g \,.\, \sin 30^\circ \,.\, l$$
$$=48^2-2 \,.\, 32 \,.\, \tfrac{1}{2} \,.\, 64$$
$$=16^2$$
$$\therefore \qquad v=16 \text{ ft. per sec.}$$

Ex. 15. *Show that the velocity with which a particle must be projected down an inclined plane of length l and height h so that the time of descent shall be the same as taken by another particle in falling freely a distance equal to the height of the plane is*

$$\frac{l^2-h^2}{l}\sqrt{\frac{g}{2h}}.$$

Sol. Let the velocity of projection be u

$$\therefore \qquad l=ut+\tfrac{1}{2}g\sin\alpha t^2$$

where
$$t=\sqrt{\frac{2h}{g}} \quad \text{and} \quad \sin\alpha=\frac{h}{l}.$$

$$\therefore \qquad ut=l-\tfrac{1}{2}g\cdot\frac{h}{l}\cdot\frac{2h}{g}=\frac{l^2-h^2}{l}.$$

$$\therefore \qquad u=\frac{l^2-h^2}{l}\cdot\frac{1}{t}=\frac{l^2-h^2}{l}\cdot\sqrt{\frac{g}{2h}}.$$

Exercise II (D)

1. Prove that the velocity acquired by a heavy particle sliding freely down a smooth inclined plane of given height is independent of the inclination of the plane.

2. Show that the time of fall down an inclined plane of given base is least when the inclination is 45°. *(P.U.)*

3. A right angled triangle is placed with its sides horizontal and vertical. Find their ratio so that the time of falling down the vertical and describing the base with the velocity acquired may be equal to the time of descent down the hypotenuse. (1:3) *(P.U.)*

4. A body sliding down a smooth inclined plane is observed to pass through equal distances, each equal to s in consecutive intervals of time t_1 and t_2. Prove that the inclination of the plane to the horizon is

$$\sin^{-1}\left[\frac{2s(t_1-t_2)}{gt_1\,t_2(t_1+t_2)}\right].$$

5. Two particles P and Q slide down a smooth inclined plane of inclination 30°, P from the highest point and Q from a point at a distance $\frac{3}{4}$ of the length of the plane from the bottom. Where will P be when Q reaches the bottom of the plane.

If P reaches the bottom of the plane in $2-\sqrt{3}$ sec. after Q, find the length of the inclined plane.

6. From an extremity of the horizontal diameter of a circle in a vertical plane, two chords are drawn subtending angles θ and 2θ at the centre. Show that if the time down the latter chord is n times as great as that down the former, then

$$n^2-1=\sec\theta.$$

7. A body begins to slide down from the top of a smooth inclined plane of height h and at the same instant another body is projected upwards from the foot of the plane with such a velocity that the bodies meet in the middle of the plane. Find the velocity of projection and the velocity of each body when they meet.

Answers

5. At $\frac{1}{4}$th of the length of the plane : $l=32$ ft.

7. $\sqrt{gh}$, 0, $\sqrt{gh}$.

Miscellaneous Exercise on Chap. II

1. A bullet is fired through three screens placed at equal intervals of a ft. and the times of passing the screens are t_1, t_2, t_3 sec. reckoned from the moment the bullet leaves the gun. Assuming that the retardation is uniform, prove that it is equal to

$$\frac{2a(t_1-2t_2+t_3)}{(t_3-t_1)(t_3-t_2)(t_2-t_1)}.$$

2. A particle is moving in a straight line with constant acceleration, show that the acceleration is

$$\frac{2\left(\frac{s'}{t'}-\frac{s}{t}\right)}{t+t'},$$ where s is the space described during any t seconds and s' during next t' seconds. (*P.U.*)

3. A particle moves in a straight line with constant acceleration and its distances from an origin O (not necessarily the position at time $t=0$) at times t_1, t_2, t_3 are x_1, x_2, x_3 respectively.

Show that if t_1, t_2, t_3 form an A.P. whose common difference is d and x_1, x_2, x_3 are in G.P. then the acceleration is

$$\frac{(\sqrt{x_1}-\sqrt{x_3})^2}{d^2}$$ assuming x_1, x_2, x_3 to be positive.

4. A particle is moving in a straight line and is observed to be at a distance a from a marked point initially, to be at a distance b after an interval of n seconds, to be at a distance c after $2n$ seconds, and at a distance d after $3n$ seconds. Prove that if the acceleration is uniform, $d-a=3(c-b)$ and that the acceleration is equal to

$$\frac{c+a-2b}{n^2}.$$

Find also the initial velocity.

5. If a point moving under uniform acceleration describes successive equal distances in times t_1, t_2, t_3, then prove that

$$\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}=\frac{1}{t_1+t_2+t_3}.$$ (*P.U.*)

6. A bullet travelling horizontally pierces successively three screens placed at equal distances a apart. If the time from the first screen to the second be t_1 and from the second to the third be t_2, prove that the retardation (assumed to be constant) is

$$\frac{2a(t_2-t_1)}{t_1t_2(t_1+t_2)}$$

and that the velocity at the middle screen is

$$\frac{a(t_1^2+t_2^2)}{t_1t_2(t_1+t_2)}. \qquad (P.U.)$$

7. Two particles P and Q move in a st. line AB towards each other. P starts from A with velocity u and acceleration f, while Q starts from B with velocity u' and acceleration f'. They pass one another at the mid-point of AB and arrive at the other ends of AB with equal velocities. Prove that

$$(u+u')(f-f')=8(fu'-uf').$$

8. A bus is beginning to move with an acceleration of 1 ft./sec². Show that

(*i*) a man who can run at 9 ft./sec., will catch the bus in 8 sec. if he is 10 ft. behind when it starts ;

(*ii*) if the man is $40\frac{1}{2}$ ft. behind he will only just catch the bus ;

(*iii*) if he is 50 ft. behind he will never catch the bus ; the shortest distance between them being $9\frac{1}{2}$ ft. (*P.U.*)

9. During the first quarter of the journey from a station A to a station B the velocity of a train is uniformly accelerated and during the last quarter it is uniformly retarded and the middle half of the journey is performed at a uniform speed. Show that the average speed of the train is two-thirds of the full speed.

10. A train stopping at two stations 5 km. apart takes 10 minutes on the journey. If the motion is first that of uniform acceleration f and then of uniform retardation f', prove that

$$\frac{1}{f}+\frac{1}{f'}=10$$

taking one kilometre as the unit of distance and one minute as the unit of time.

11. A stone is dropped from an aeroplane which is rising with acceleration f and t seconds after this another stone is dropped. Prove that the distance between the stones at a time T after the second stone is dropped is

$$\tfrac{1}{2}(g+f)(t+2T)t.$$

12. A, B, C, D are four points in a vertical line, situated at equal distances, so that AB=BC=CD. A body falls from rest at A, prove that the times of describing AB, BC, CD are as

$$1 : \sqrt{2}-1 : \sqrt{3}-\sqrt{2}.$$

13. Find the position of a point on the circumference of a vertical circle such that the time of descent from it down the radius to the centre shall be the same as the time of descent from it down to the lowest point of the circle. (*P.U.*)

14. A train travelling at 60 m.p.h. has to slow down on account of work being done on the line. Instead of continuing at a constant speed, it, therefore, moves with constant retardation of 1 m.p.h. per second until its speed is reduced to 15 m.p.h. It then travels at a constant speed for $\frac{1}{4}$ mile and then accelerates at $\frac{1}{2}$ mile per hour per second until its speed is once again 60 m.p.h. Find the delay caused.

15. A particle moves from rest with a uniform acceleration. Prove that the average velocity is half or two-thirds of the final velocity, according as the time or the space is divided into an infinite number of equal parts and average taken with regard to these.

16. A particle takes t sec. less and acquires a velocity of s ft./sec. more at one place than at another in falling from rest through the same distance. Show that the gravities g_1 and g_2 at the two places are related by $g_1g_2=s^2/t^2$.

Answers

4. $u=\dfrac{4b-c-3a}{2n}$.

13. The radius through the point makes 60° with the vertical.

14. 1 sec.

CHAPTER III

KINETICS—FORCE AND MOTION

(A) Newton's Laws of Motion

3·1 We have dealt with, in chapter II, the geometry of motion of a particle in a straight line without any reference to the cause of the motion. We shall now take into account this factor in studying the motion. This branch of the subject is called *Kinetics*.

3·11 Definitions : The *Mass* of a body is the quantity of matter in the body.

The *Density* of a uniform body is the mass of a unit volume of the body. Thus if M is the mass of volume V of a body whose density is ρ, then

$$\rho=\frac{M}{V}.$$

The British unit of mass is the ***Imperial Pound*** which consists of a lump of platinum kept in Westminster.

The French unit of mass is called a *gramme* which is the one-thousandth part of a certain quantity of platinum kept in Pairs.

(One pound=453·6 grammes nearly)

This system has also been adopted by the Government of India recently.

The *Momentum* of a body is the product of its mass and velocity. It has the same direction and sense as the velocity of the body. Thus the momentum is a *vector quantity*. If m be the mass of a body moving with a velocity v, its

momentum=mv.

3·2 Newton's Laws of Motion : Newton, in his ***Principia***, stated three laws of motion, on which is based the subject of Dynamics.

Law I. Everybody continues to be in its state of rest, or of uniform motion in a straight line, except in so far as it is compelled by some external impressed force to change that state.

Law II. The rate of change of momentum is proportional to the impressed force, and takes place in the direction in which the force acts.

Law III. To every action there is an equal and opposite reaction, *i e* the actions and reactions are always equal and opposite.

3·21 The First Law : One of the two statements contained in this law is that a material body at rest remains at rest provided it is not acted on by some external force. This is self evident because a book on a table remains there without any movement unless it is lifted or pushed or the table is moved to some other position.

The second statement is that a body in motion remains in uniform motion for ever unless it is acted on by some external force. This part of the first law is somewhat difficult to realise in practice. A ball rolling on the floor does not continue to move for ever but gradually comes to rest. This may prove the law to be wrong but a little consideration shows that the motion of the ball is constantly obstructed by the friction of the floor—which is the external force. If it is possible to eliminate the friction of the floor completely, the ball will continue to move on the floor for ever.

The first law states a principle called the *Principle of Inertia.* Inertia is that property of material bodies by virtue of which they are unable, of their own accord, to change their state of rest or of uniform motion.

3·22 Force : According to Newton's First Law of motion, force is necessary (*i*) to change the state of rest of a body or to bring the body into motion, (*ii*) to change the speed of a body moving in a straight line, and (*iii*) to change the direction of a body without even changing its speed. This enables us to form a definition of force.

Force : *A force is that which changes or tends to change the state of rest or of uniform motion of a body in a straight line.*

The *impressed force* means the external force acting on the body.

3 23 The Second Law : The first part of this law is that the rate of change of momentum is proportional to the impressed force. This part of the law gives us a method of measuring force.

Let m denote the mass of a body, v its velocity. Let P be the force applied to the body and f the acceleration produced. Then since P is proportional to the rate of change of momentum,

$$P \propto \frac{d}{dt}(mv)$$

or

$$P = k\frac{d}{dt}(mv), \text{ where } k \text{ is a constant,}$$

$$= km\frac{dv}{dt}, \ m \text{ being considered as constant,}$$

i.e.

$$P = kmf.$$

Let us choose the unit of force in such a way that unit force acting on unit mass produces a unit acceleration, *i.e.* $P = 1$,

when $m = 1$ and $f = 1$.

Thus $k = 1$

Hence $P = mf.$

This equation is the fundamental equation of Dynamics. It is known as *equation of motion.*

This is sometimes stated in words as

$$\text{Acceleration}=\frac{\text{Force producing motion}}{\text{Mass moved}}.$$

If $$P=0,$$

then $$mf=0$$

or $$m\frac{dv}{dt}=0$$

or $$\frac{d}{dt}(mv)=0$$

i.e. momentum, $mv=$constant.

Thus when the body is not acted upon by an impressed force, it continues to move with constant momentum. Thus Law I is a particular case of Law II.

3·3 Units of force : *The unit of force is that force which acting on unit mass generates in it unit acceleration.*

In F.P.S. (or English) system the unit of force is the *Poundal* which is *a force that will create in a mass of one pound an acceleration of one foot per second per second.*

In C.G.S. (or French) system the unit of force is the *Dyne* which is *a force that will create in a mass of one gramme an acceleration of one centimetre per second per second.*

These are called *Absolute Units* of force because their values are same everywhere, being independent of the acceleration due to gravity.

3·31 Weight : It is defined as the force which the earth exerts on the body. Since acceleration produced by the attraction of the earth is g, the weight W of the body of mass m is given by

$$W=mg$$

in absolute units.

If $m=1$ lb., weight$=g$ poundals.

Thus the weight of one pound is equal to a force of $g(32)$ *poundals*.

If $m=1$ gramme, weight$=g$ dynes.

Thus the weight of one gramme is equal to a force of $g(981)$ *dynes.*

The weight of one pound is called a pound weight (or simply a pound) and the weight of one gramme is called a gramme-weight (or simply a gramme).

These are the *Gravitational Units* of force.

Thus 1 pound weight (or lb. wt.) $=g$ (32) poundals.

1 gramme weight (or gm. wt.)$=g$ (981) dynes.

3·32 Value of g varies slightly from 32 to 32·2 in F.P.S. system at different places on the surface of the earth. Thus the value of Gravitational Unit of Force is not same at all places.

3·33 Distinction between mass and weight : The mass of a body being the quantity of matter present in the body is a constant quantity whereas the weight of a body is the force with which it is attracted to the centre of the earth and since the acceleration due to gravity varies at different places, the weight of a body changes from place to place.

3·34 The weight of a body is proportional to its mass : Let W_1 and W_2 be the weights of two bodies whose masses are m_1 and m_2 respectively. Then at the same place

$$W_1 = m_1 g$$

$$W_2 = m_2 g$$

$$\therefore \quad \frac{W_1}{W_2} = \frac{m_1}{m_2}.$$

Thus the weight of a body is proportional to its mass.

3·35 Physical Independence of Forces : The second part of Newton's second law of motion states that the rate of change of momentum produced by the impressed force is in the direction in which the force acts, that is, the acceleration produced by a force is in the direction of the force. If two or more forces act on a body at the same time, each produces an acceleration proportional to it in its own line of action independent of the motion of the body. This leads to a principle called the principle of *Physical Independence of Force* which states that if *several forces act on a body simultaneously, then the effect of each force is the same as if the others were absent.*

3·4 The Third Law : According to this law, (1) an action is always accompanied by a reaction and (2) the action and the reaction are always equal and opposite.

No force in this universe exists singly. For every force in nature there must be another force equal and opposite to it in the same straight line These two together are termed *Stress* between the bodies.

Considering one portion of a body, the effect on it on the other is called *Action*, the opposite force on the other portion is called *Reaction.*

Illustrations : (1) If a man presses the table with his finger, his finger also is pressed by the table with a force equal and opposite to that which the finger exerts on the table.

(2) **Horse and cart** : In this case the force of tension with which the horse pulls the cart forward is equal and opposite to the tension with which the cart pulls the horse backwards. But since the force of tension is not an external force, it cannot affect the system. The external force affecting motion is the friction between the feet of the horse and the ground. The horse pushes the ground backwards and an equal amount of friction, F say, acts forwards.

A small amount of friction, F′ say, also acts on the cart checking its motion. The external force F−F′ makes the cart move.

Ex. 1. *Find the magnitude of the force, which acting on a mass of 2 lb., produces in it an acceleration of 5 ft./sec². What will be the velocity of the body after 20 seconds ?*

Sol. Here

$$m=2 \text{ lb.}, \quad f=5 \text{ ft./sec}^2.$$

$$\begin{aligned} P &= mf \\ &= 2\times5 \text{ poundals} \\ &= \tfrac{10}{32} \text{ lb. wt.} \end{aligned}$$

Let v denote the required velocity.

Then

$$v=ft=5\times20=100 \text{ ft./sec.}$$

Ex. 2. *A force equal to the weight of 5 lb. acts on a mass of 16 lb. for 2 seconds ; find the distance it has passed over. If at the end of 2nd second the force ceases to act, how far will the body travel in the next 10 seconds if subjected to a resistance of 1·5 lb. wt. ?*

Sol.

$$\begin{aligned} P &= 5 \text{ lb. wt.} \\ &= 5\times32=160 \text{ poundals.} \end{aligned}$$

Let f be the acceleration. Then

$$f=\frac{P}{m}=\frac{160}{16}=10 \text{ ft./sec}^2.$$

Let s denote required distance, then

$$\begin{aligned} s &= \tfrac{1}{2}ft^2=\tfrac{1}{2}\times10\times2^2 \\ &= 20 \text{ ft.} \end{aligned}$$

The velocity acquired in 2 seconds is

$$v=ft=10\times2=20 \text{ ft./sec.}$$

The force ceases to act at the end of 2nd second and resistance acts during the subsequent part of the motion. Let r be the retardation produced by the resisting force, so that

$$\begin{aligned} r &= \frac{1{\cdot}5\times32}{16} \\ &= 3 \text{ ft./sec}^2. \end{aligned}$$

$$\begin{aligned} \therefore \quad S' &= 20\times10-\tfrac{1}{2}\times3\times10^2 \\ &= 50 \text{ feet.} \end{aligned}$$

Ex. 3. *A mass of 10 lb. falls 10 ft. from rest and is then brought to rest by penetrating 1 foot into a layer of sand. Find the average thrust on it.*

Sol. Let v be the velocity acquired in falling through 10 feet. Then

$$\begin{aligned} v^2 &= 2gh=2\times32\times10 \\ &= 640 \text{ ft./sec.} \end{aligned}$$

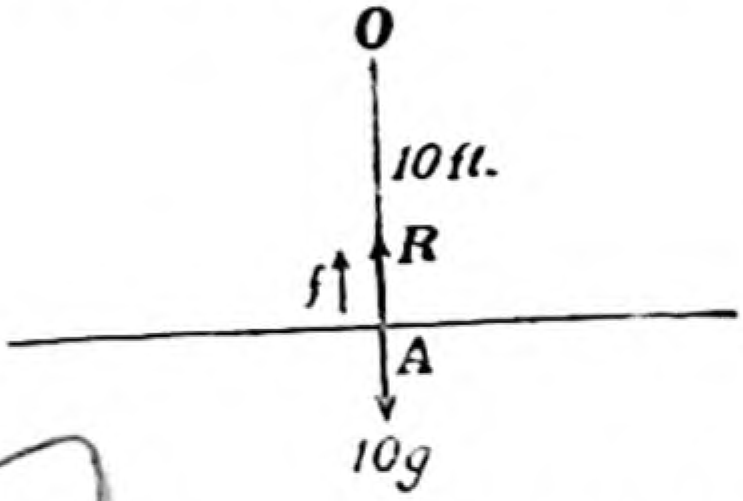
The sand offers resistance to the motion and a retardation is experienced by the body.

The velocity is zero when the mass penetrates 1 foot into the sand. The retardation is given by

$$0=640-2\times 1\times f \quad (v^2=u^2-2fs)$$

or $$f=320 \text{ ft./sec}^2.$$

Let R be the resistance of the sand. Then because the weight of the mass is acting in the downward vertical direction.



$$mf=R-mg$$

or $$R=m(f+g)$$

$$=10(320+32)$$

$$=35\ 0 \text{ poundals}$$

$$=\tfrac{3520}{32}=110 \text{ lb. wt.}$$ **Ans.**

Ex. 4. *An engine of mass 30 tons pulls after it a train of 130 tons; supposing the friction to be $\frac{1}{50}$th of the weight of the whole train calculate the force exerted by the engine, if at the end of first mile from the start the speed be raised to 45 m.p h.*

(P.U. T.D.C. I, 1966 ; T.D.C. II, 1955)

Sol. Total mass of the train $=160$ tons
$=160\times 2240$ lb.

Now
$u=0,$
$v=45$ miles/hr.
$=66$ ft./sec.
$s=1$ mile
$=3\times 1760$ ft.

The acceleration of the train is given by

$$v^2=u^2+2fs$$
$$66^2=0+2f\times 1760\times 3$$
$$f=\tfrac{23}{80} \text{ ft./sec}^2.$$

$\therefore$

Accelerating force is

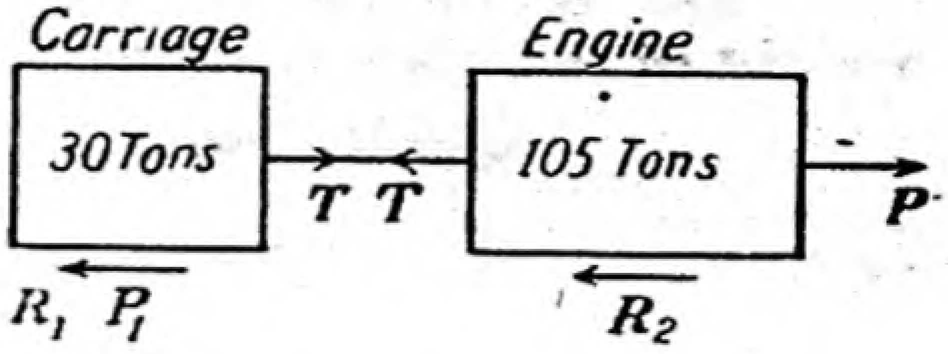
$$P=mf$$
$$=\frac{160\times 2240\times 30}{32\times 80} \text{ lb. wt.}$$
$$=4620 \text{ lb. wt.}$$

Force of friction $$=\frac{160\times 2240}{50}$$
$$=7168 \text{ lb. wt.}$$

$\therefore$ Pull of the engine $=4620+7168$
$=11788$ lb. wt.

$$=\frac{11788}{2240}=5\tfrac{21}{80} \text{ tons wt.}$$

Ex. 5. *An engine of mass 105 tons in coupled to and pulls a carriage of mass 30 tons ; the resistance to the motion of the engine is $\frac{1}{100}$ of its weight ; the resistance to the motion of the carriage is $\frac{1}{150}$ of its weight. Find the tension in the coupling if the whole tractive force exerted by the engine is equal to the weight of 6000 lb.*



Sol. P, the tractive force
$=6000$ lb. wt.
Total resistance
$=(\frac{105}{100}+\frac{30}{150})\times 2240$ lb wt.
$=2800$ lb. wt.
∴ the accelerating force
$=6000-2800$
$=3200$ lb. wt.

m, the total mass $=135\times 2240$ lb.

$$\therefore\ f,\text{ the acceleration }=\frac{3200\times 32}{135\times 2240}=\frac{64}{189}\text{ ft./sec}^2.$$

To find the tension in the coupling T, consider the motion of the carriage alone.

The accelerating force on the carriage

$$=\text{mass of the carriage}\times\text{acceleration}$$

$$=\frac{30\times 2240\times 64}{189}\text{ poundals}$$

$$=\frac{22400\times 64}{63\times 3}\text{ lb. wt.}$$

$$=711\tfrac{1}{9}\text{ lb. wt.}$$

R_1, the resistance to the carriage

$$=\frac{30\times 2240}{150}=448\text{ lb. wt.}$$

Accelerating force $=$ Tension in the coupling $-$ resistance
$=T-R_1$

∴ T, the tension in the coupling

$$=(711\tfrac{1}{9}+448)\text{ lb. wt.}$$
$$=1159\tfrac{1}{9}\text{ lbs. wt.}$$

Note. T can also be found by considering the motion of the engine alone.

In this case, we have

$$P-T-R_2=mf$$

or $\quad 6000\times 32-T-\frac{105}{100}\times 2240\times 32=105\times 2240\times\frac{64}{189}$

which gives T as before.

Exercise III (A)

1. What force in pounds weight will give a truck of 9 tons a velocity of 30 m.p.h. in one minute ?

2. A railway truck weighing 7 tons is pulled by a constant

force and travels 50 ft. from rest in 10 sec. Find the magnitude of the force.

3. A train weighing 110 tons is running at 15 miles per hour when steam is shut off and a resistance of 10 lb. per ton acts continuously until it is brought to rest. Find the distance it travels after steam is shut off, and the time taken to come to rest.

✱ A ball of mass 8 oz. falls freely under gravity through a distance of 16 ft. In the process of catching a man allows his hands to drop a distance 1 ft. Find the average pressure on his hands during the catch.

5. A bullet weighing 100 grammes and moving horizontally at the rate of 200 cm. per second is fired into a vertical log of wood into which it penetrates 10 cm. If it is fired into the similar piece of wood 5 cm. thick, with what velocity would it emerge? Find also the force of resistance, supposing it to be uniform.

6. A mass m is acted on by a constant force of P lb. wt. under which in t sec. it moves a distance x feet, and acquires a velocity v ft. per sec. Show that

$$x=\frac{gt^2\text{P}}{2m}=\frac{v^2m}{2g\text{P}}$$ *(P.U. T.D.C. II, 1965, 63)*

7. If P be the tractive force in tons on a train weighing W tons and R tons the resistance, prove that the least time of travelling a distance s from rest to rest is

$$\left[\frac{2s}{g}\cdot\frac{\text{WP}}{\text{R(P}-\text{R)}}\right]^{\frac{1}{2}},$$ *(P.U. T.D.C. II, 1966)*

and the maximum velocity is

$$\left[\frac{2gs\ \text{R(P}-\text{R)}}{\text{WP}}\right]^{\frac{1}{2}}.$$ *(P.U.)*

8. An engine which exerts a constant pull is attached to a train of mass 120 tons (including the engine), the resistance due to friction etc., being 14 lb. per ton. It runs for a mile on a level track and gets up a velocity of 30 miles an hour.

(*i*) Find the force exerted by the engine.

(*ii*) How far will the train run up an incline of 1 in 100, steam being shut off?

9. A train of 300 tons drawn up by an engine weighing 100 tons starts from rest with constant acceleration and acquires a velocity of 30 m.p.h. on the level in one minute. Find the acceleration and the force exerted by the engine.

Find also the force of tension in the coupling between the engine and the rest of the train. *(P.U. Engg. 1966)*

10. A train of mass 160 tons starts from a station, the engine exerting a tractive force of $2\frac{1}{2}$ tons in excess of the resistances until a speed of $37\frac{1}{2}$ m.p.h. is attained. This speed continues constant until the brakes, causing a retardation of $2\frac{1}{2}$ ft./sec², bring the train to rest 5 miles away. Find the time taken (*i*) during acceleration ; (*ii*) during retardation ; (*iii*) altogether.

✓ **11.** A small balloon of total mass 10 lb. at a height of 100 ft. above the ground, is moving upward with a velocity of 60 f.p.s. and acceleration 4 f.p.s.s., when a mass of 1 lb. becomes detached. Find the new acceleration of the balloon, assuming that the upward force acting upon it remains unaltered.

Find how many seconds it takes the 1 lb. mass to reach the ground, neglecting air resistance. (*Lucknow*)

✗ **12.** A train of mass 105 tons is travelling with a uniform speed, the resistance being 10 lb. wt. per ton. If the last bogie weighing 15 tons gets detached how much will the other portion have gained on it in 48 seconds assuming the pull of the engine and resistance per ton be same as before ?

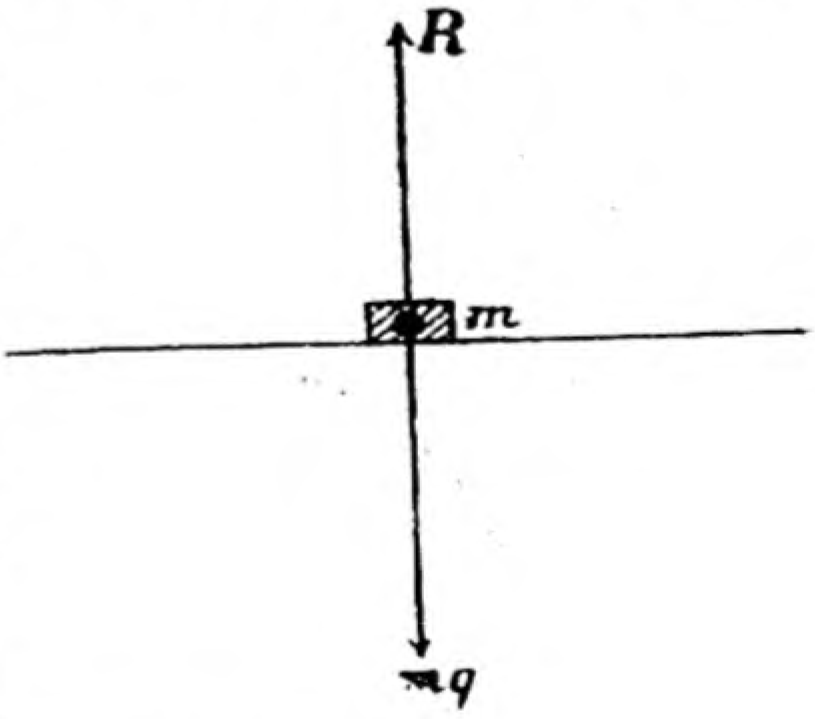
Answers

1. 452 lb. wt. **2.** 490 lb. wt. **3.** 1694 ft. ; 154 sec.
4. 8·5 lb. wt. **5.** $100\sqrt{2}$ cm.p s. ; $222\frac{2}{9}$ gm. wt.
8. 3220 lb. wt. ; $1861\frac{7}{13}$ ft.
9. $f=\frac{11}{15}$ f.p.s.s. ; $P=9\frac{1}{6}$ ton wt. ; $T=6\frac{7}{8}$ ton wt.
10. 110 sec. ; 22 sec. ; 546 sec. **11.** $f=8$ f.p.s.s. ; 5 sec.
12. 192 feet.

(B) MOTION OF CONNECTED PARTICLES

3·5 Weight carried by a lift : *A body of mass m lb., is placed on a horizontal plane which is moving with an upward vertical acceleration f : to find pressure between the body and the plane.*

The forces acting on the mass are the reaction R of the plane of the lift vertically upwards and its weight mg acting vertically downwards. The resultant upward force acting on the mass is $R-mg$.



∴ equation of motion (by Newton's second law of motion) is

$$mf = R - mg$$

$$\therefore \quad R = m(g+f) \qquad \ldots(1)$$

3 51 Thus the pressure is greater than weight of the body by an amount mf.

If the plane be moving downwards with an acceleration f, the resultant force downwards is $mg-R$ so that the equation of motion now is

$$mf = mg - R$$

$$\therefore \quad R = m(g-f) \qquad \ldots(2)$$

i.e. the pressure is less than the weight of the body by an amount mf.

In (2), if $f=g$, $R=0$, *i.e.* there is no pressure between the body and the plane so that the body and the plane move freely.

Again, in (2), if $f>g$, R is negative so that plane pulls the mass instead of pushing it which is impossible. The body is left behind and moves freely.

Ex. 6. *A mass of 10 lb. is hung on a spring balance which is suspended from the roof of a lift. What will the readings of the balance be when the lift is*

(*i*) *rising with a uniform velocity of 6 ft./sec.,*

(*ii*) *rising with an acceleration of 3 ft./sec^2,*

(*iii*) *descending with an acceleration of 3 ft./sec^2.*

Sol. (*i*) Since the velocity is uniform, $f=0$.

$\therefore$ reading of the spring balance = real wt.

$$= 10 \text{ lb. wt.}$$

(*ii*) Let R be the reading of the spring balance. Since the lift is rising with an acceleration of 3 ft./sec^2,

$$\begin{aligned} R &= 10(32+3) = 350 \text{ poundals} \\ &= \tfrac{350}{32} \text{ lb. wt.} \\ &= 10\tfrac{15}{16} \text{ lb. wt.} \end{aligned}$$

(*iii*) Let R′ be the reading of the spring balance. Since the lift is descending with an acceleration of 3 ft./sec^2,

$$\begin{aligned} R' &= 10(32-3) = 290 \text{ poundals} \\ &= \tfrac{290}{32} \text{ lb. wt.} \\ &= 9\tfrac{1}{16} \text{ lb. wt.} \end{aligned}$$

Exercise III (B)

1. A weight of 12 kilogram is placed on the floor of a lift which is made to ascend with

(*i*) a uniform acceleration of 3 cm. p.s.s.

(*ii*) a uniform velocity of 240 cm. p.s.

Find the pressure on the floor in each case.

2. A 4 lb. weight carried on a spring balance in a balloon has an apparent weight of 4·8 lb. when the balloon is ascending. What is the acceleration of the balloon? What should the body weigh if the balloon is descending (*i*) with an acceleration of 10 ft. per sec.^{2}, (*ii*) with uniform velocity of 20 miles per hour, (*iii*) freely.

3. A parachute, weighing 1 cwt. falling with a uniform acceleration from rest, describes 16 ft. in the first 4 seconds. Find the resultant vertical pressure of the air on the parachute.

4. A body whose true weight is 13 oz. appeared to weigh 12 oz. when weighed by means of a spring balance in a moving lift. What was the acceleration of the lift at the time of weighing? (*Delhi*).

5. A lift weighing 1500 lb. is pulled by a rope vertically upwards with an acceleration of 4 ft./sec^2. What is the tension of the rope? What will it be when the lift is being lowered with the same acceleration?

6. A lift whose weight is 19 cwt. is raised from rest by a force of one ton, the force being removed at a time such that the lift comes to rest again after moving through a total height of 20 feet. Prove that the total time from rest to rest is 5 seconds. *(Panjab)*

7. A cage of $\frac{1}{4}$ ton is being raised from rest with a uniform acceleration by a cable. The cable is allowed to become slack and the load comes to rest 225 feet above its original position. The total time form rest to rest being $11\frac{1}{4}$ seconds, find the tension at the lower end of the cable when it is tight, the weight of the cable being neglected.

8. A diver jumps into a well with a life belt of weight 4 kilogram tied on to his head. What pressure does the belt exert on his head before he reaches the surface of water.

Answers

1. (*i*) R$=$12·04 kg. wt. approx. (*ii*) R$=$12 kg. wt.
2. $f=6{\cdot}4$ ft. p.s.s. ; $2\frac{3}{4}$ lb. ; 4 lb. ; zero.
3. 105 lb. wt. 4. $2\frac{6}{13}$ f.p.s.s.
5. 1687·5 lb. wt. ; 1312·5 lb. wt.
7. T$=$630 lb. wt. 8. Nil.

3·6 Motion of two particles connected by a string :

Two particles, of masses m_1 and m_2, $(m_1>m_2)$ are connected by a light inextensible string which passes over a small smooth fixed pulley. To find the resulting motion of the system, and the tension in the string.

Since the pulley is smooth, the tension T of the string is the same throughout the string.

Let at any instant, x and y be the depths of the masses m_1 and m_2 below the pulley. If l be the length of the string,

$$x+y=l.$$

Differentiating both sides with respect to t,

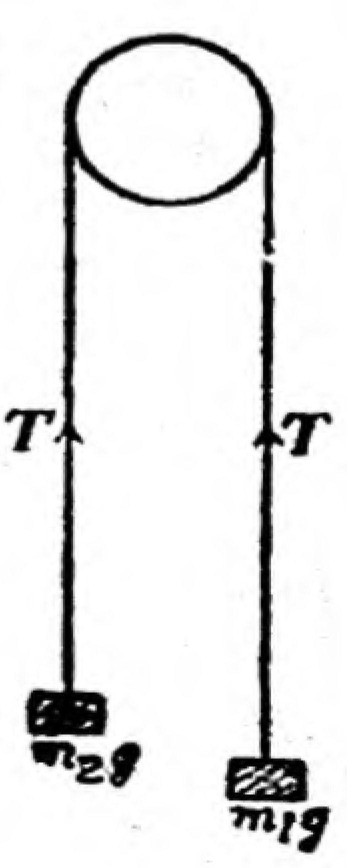
$$\frac{dx}{dt}+\frac{dy}{dt}=0$$

or

$$\frac{dx}{dt}=-\frac{dy}{dt},$$

Differentiating again,

$$\frac{d^2x}{dt^2}=-\frac{d^2y}{dt^2}.$$



Thus the velocity and the acceleration of the mass m_1 downwards is equal to the velocity and the acceleration of the mass m_2 upwards.

Let f be the common acceleration. The equations of motion of mass m_1 and m_2 are respectively

$$m_1 f = m_1 g - \mathrm{T}$$

$$m_2 f = \mathrm{T} - m_2 g.$$

Solving these equations for f and T, we get

$$f = \frac{m_1 - m_2}{m_1 + m_2} g$$

and
$$\mathrm{T} = \frac{2m_1 m_2}{m_1 + m_2} g.$$

The above equation gives

$$\frac{2}{\mathrm{T}} = \frac{m_1 + m_2}{m_1 m_2 g}$$

$$= \frac{1}{m_1 g} + \frac{1}{m_2 g}.$$

Thus the tension in the string is equal to a weight which is the harmonic mean between the two given weights.

3·61 Pressure on the pulley : The forces acting on the pulley are two equal tensions T acting along the string on both of its sides. If R be the pressure on the pulley,

$$\mathrm{R} = 2\mathrm{T} = \frac{4m_1 m_2}{m_1 + m_2} g.$$

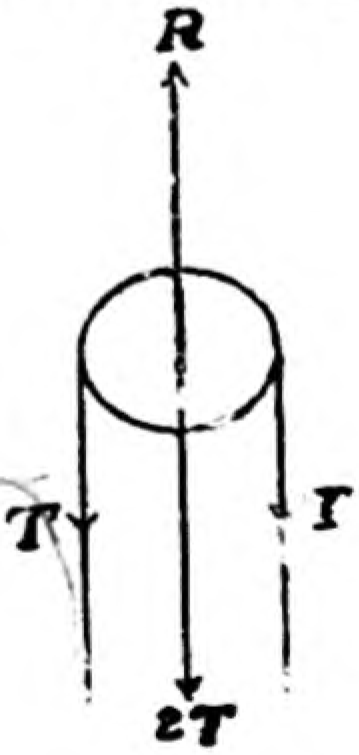
The pressure on the pulley is always less than the sum of the weights of the masses.

For

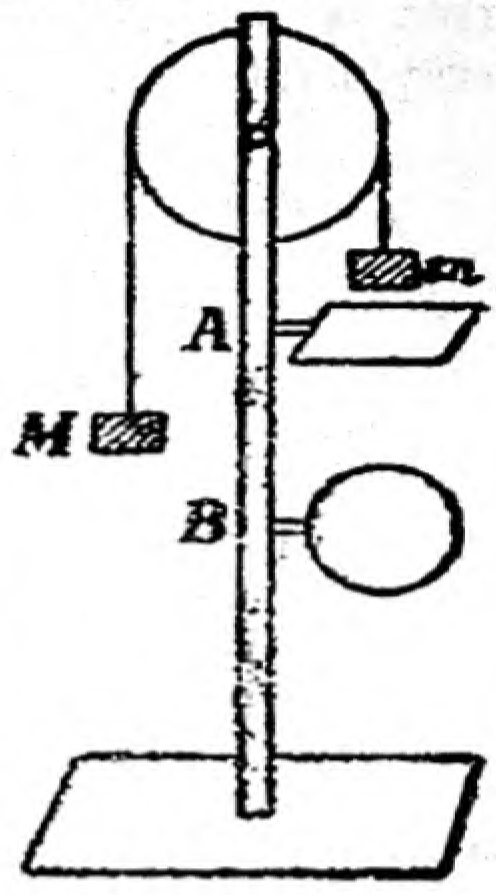
$$(m_1 + m_2) g - \mathrm{R} = (m_1 + m_2) g - \frac{4m_1 m_2}{m_1 + m_2} g,$$

$$= \frac{(m_1 - m_2)^2}{m_1 + m_2}.$$

which is always positive.



3·7 Atwood's Machine : This machine is used to find the approximate value of g. Two equal weights M each, are attached to the ends of a string passing over a light frictionless pulley. An additional weight m (called the rider), resting on the platform A, is put on one of the weights. The platform A is dropped and the system begins to move with an acceleration



$$f=\frac{(m+M)-M}{(m+M)+M}g=\frac{m}{m+2M}g.$$

Let v be the velocity acquired when it has moved a distance $AB=s$.

Then $\quad v^2=2fs.$

As soon as the weight carrying the rider passes through the ring B, the rider is caught off so that the weights on the two sides become equal and the system now moves with a constant velocity v. The distance h moved with this velocity and t, the time taken, are measured. Then since

$$h=vt,$$

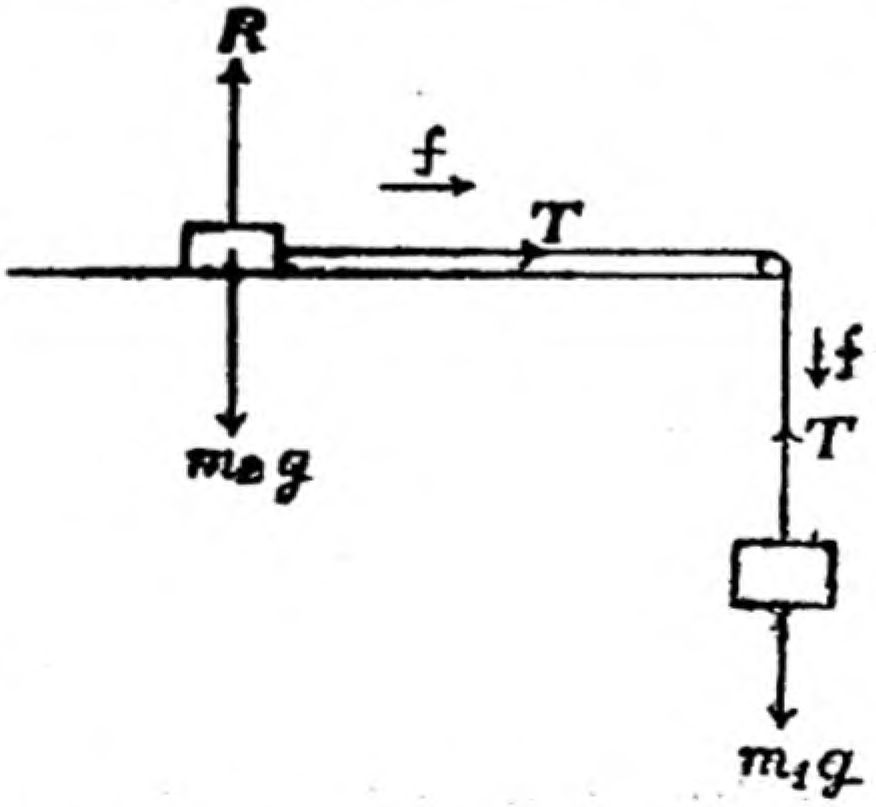
we have

$$h^2=t^2v^2=t^2.2\,fs=2st^2.\frac{m}{m+2M}g,$$

which gives an approximate value of g.

3·8 Motion on a Smooth Table : *Two particles of massss m_1 and m_2 are connected by a light inextensible string ; m_2 is placed on a smooth horizontal table, the string passes over a light pulley at the edge of the table and m_1 is hanging freely ; to find the resulting motion and the tension of the string.*

Let, at any time, x and y be the distances of the masses from the edge of the table.



Then $x+y=l$, where l is the length of string.

Differentiating both sides with respect to t,

$$\frac{dx}{dt}+\frac{dy}{dt}=0$$

$$\therefore\quad \frac{dx}{dt}=-\frac{dy}{dt}.$$

Similarly,

$$\frac{d^2x}{dt^2}=-\frac{d^2y}{dt^2}.$$

Thus the velocity and acceleration of m_1 away from the edge (*i.e.* vertically downwards) are equal to the velocity and acceleration of m_2 towards the edge.

The pulley being smooth, the tension T of the string is the same throughout its length.

The equation of motion for m_1 is

$$m_1 f = m_1 g - T \qquad \text{..(1)}$$

The equation of motion for m_2 in the horizontal direction is

$$m_2 f = T \qquad \text{...(2)}$$

(Since the mass m_2 does not move perpendicular to the table, $R = m_2 g$).

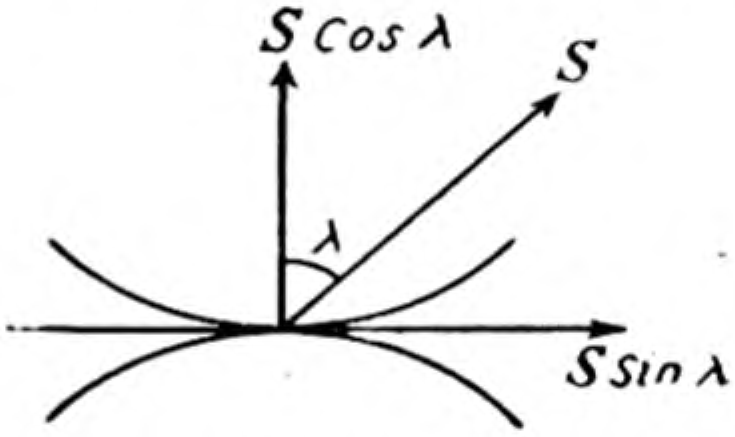
Solving (1) and (2), we get

$$f = \frac{m_1}{m_1 + m_2} g \quad \text{and} \quad T = \frac{m_1 m_2}{m_1 + m_2} g.$$

3·81 Pressure on the Pulley : The pressure on the pulley is the resultant of two equal forces T at right angles.

$$\therefore \quad R, \text{ the pressure on the pulley} = \sqrt{T^2 + T^2}$$
$$= \sqrt{2}T$$
$$= \sqrt{2}\,\frac{m_1 m_2}{m_1 + m_2}\,g.$$

3·82 Friction : When two perfectly smooth bodies are in contact, the direction of reaction between them is along the common normal. But when the bodies are rough, the direction of reaction between them is not along the common normal In this case. let λ be the direction of reaction with the common normal. Then resolved parts of S, the resultant reaction, along the common normal and along the tangent line are S cos λ and S sin λ respectively. The former is called normal reaction and the later is called friction. It is this friction which opposes motion.



Whenever a rough body, in contact with another rough body, is in motion, force of friction always acts in the direction opposite to the direction of motion.

There are three kinds of friction :

Statical Friction : When two bodies in contact are in equilibrium, the friction exerted is just sufficient to maintain equilibrium and is called *Statical friction.*

Limiting Friction : When one body is just on the point of sliding on another, the friction exerted is called *Limiting friction*. Friction in this stage is of maximum value.

Dynamical Friction : When one body moves or slides on another body, the friction exerted is called *Dynamical friction*.

The magnitude of the limiting friction (or dynamical friction) bears a constant ratio to the normal reaction. Thus if F be the limiting friction (or dynamical friction) and R the normal reaction at the point of contact, then

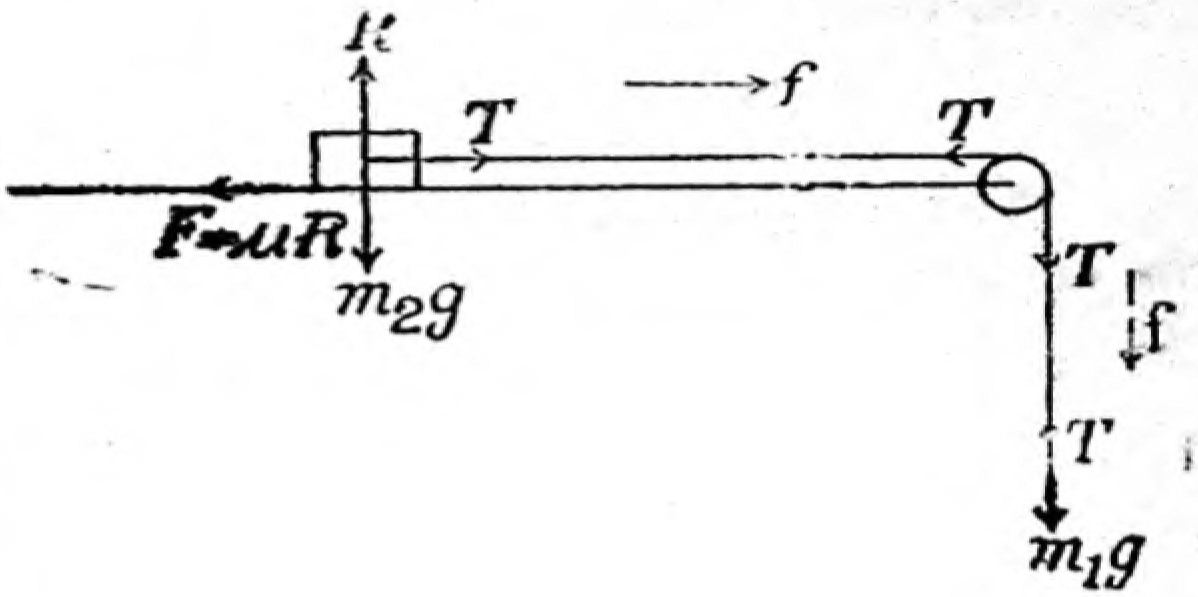
$$\frac{F}{R} = \mu,$$

where μ is a constant called the co-efficient of friction.

$$\therefore \qquad F = \mu R.$$

3 83. **Motion on a Rough Table :** *A mass m_2 is placed on a rough horizontal table, and connected by a light inextensible string passing over a smooth pulley at the edge to a mass m_1, hanging freely ; to find the resulting motion and the tension in the string.*

The force of friction μR acts on mass m_2 in the direction opposite to T.



The equations of motion are

$$m_1 f = m_1 g - T \qquad \ldots (1)$$

$$m_2 f = T - \mu R \qquad \ldots (2)$$

$$R = m_2 g \qquad \ldots (3)$$

From (2) and (3),

$$m_2 f = T - \mu m_2 g \qquad \ldots (4)$$

Solving (1) and (4), we get

$$f = \frac{m_1 - \mu m_2}{m_1 + m_2} g$$

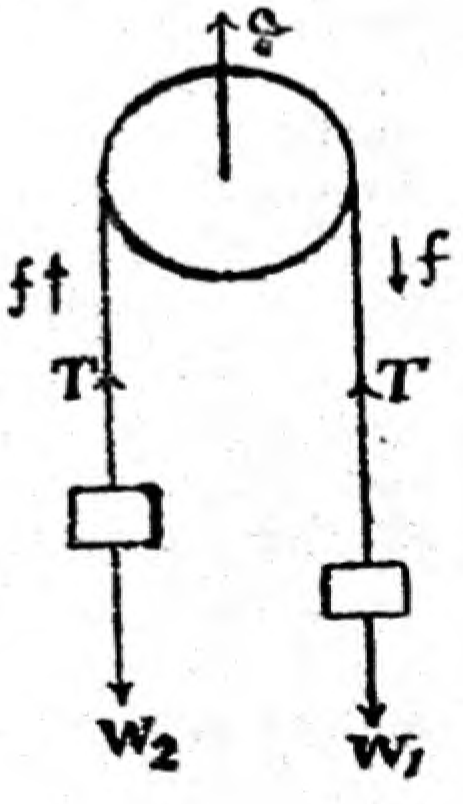
and

$$T = \frac{m_1 m_2 (1+\mu)}{m_1 + m_2} g.$$

The body moves if $m_1 > \mu m_2$.

Ex. 7. *Two weights W and W′ are connected by a light string passing over a light pulley. If the pulley moves upwards with an acceleration equal to that of gravity, show that the tension of the string is*

$$\frac{4WW'}{W+W'}.$$



Sol. Let m_1 and m_2 be the masses of the two weights.

Then

$$m_1 = \frac{W}{g}$$

and

$$m_2 = \frac{W'}{g}.$$

Let f be the common acceleration of the two masses if the pulley were at rest. Due to upward acceleration of the pulley, the acceleration of m_1 is $(f-g)$ vertically downwards and that of m_2 is $(f+g)$ upwards.

The equations of motion for the two masses are

$$m_1 (f-g) = m_1 g - T$$

and

$$m_2 (f+g) = T - m_2 g.$$

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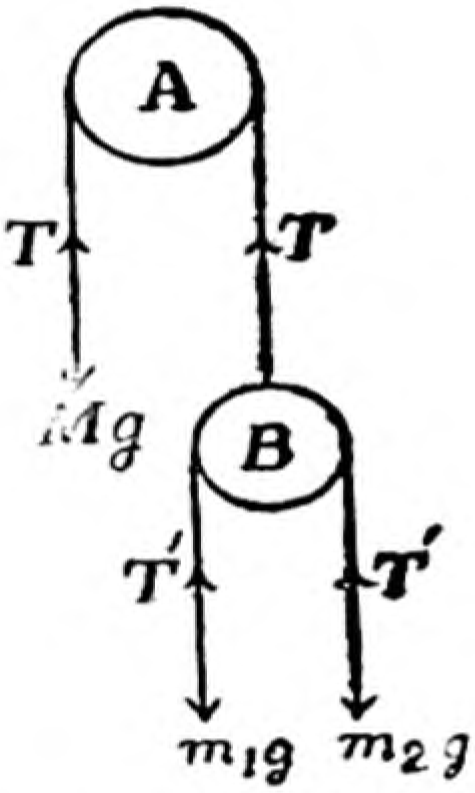
Solving for T, we get

$$T=\frac{4m_1m_2}{m_1+m_2}g=\frac{4WW'}{W+W'}$$

Ex. 8. *A string passes over a smooth fixed pulley and to one end there is attached a mass M, and to the other a smooth light pulley over which passes another string with masess m_1 and m_2 at the ends. If the system is released from rest, show that M will not move if*

$$\frac{4}{M}=\frac{1}{m_1}+\frac{1}{m_2}.$$

(Panjab)



Sol. Since the pulley A does not move

$$\therefore \quad T=Mg.$$

The pressure at the pulley B

$$=\frac{4m_1m_2}{m_1+m_2}g$$

This pressure must be equal to T

$$\therefore \quad T=\frac{4m_1m_2}{m_1+m_2}g$$

or

$$Mg=\frac{4m_1m_2}{m_1+m_2}g$$

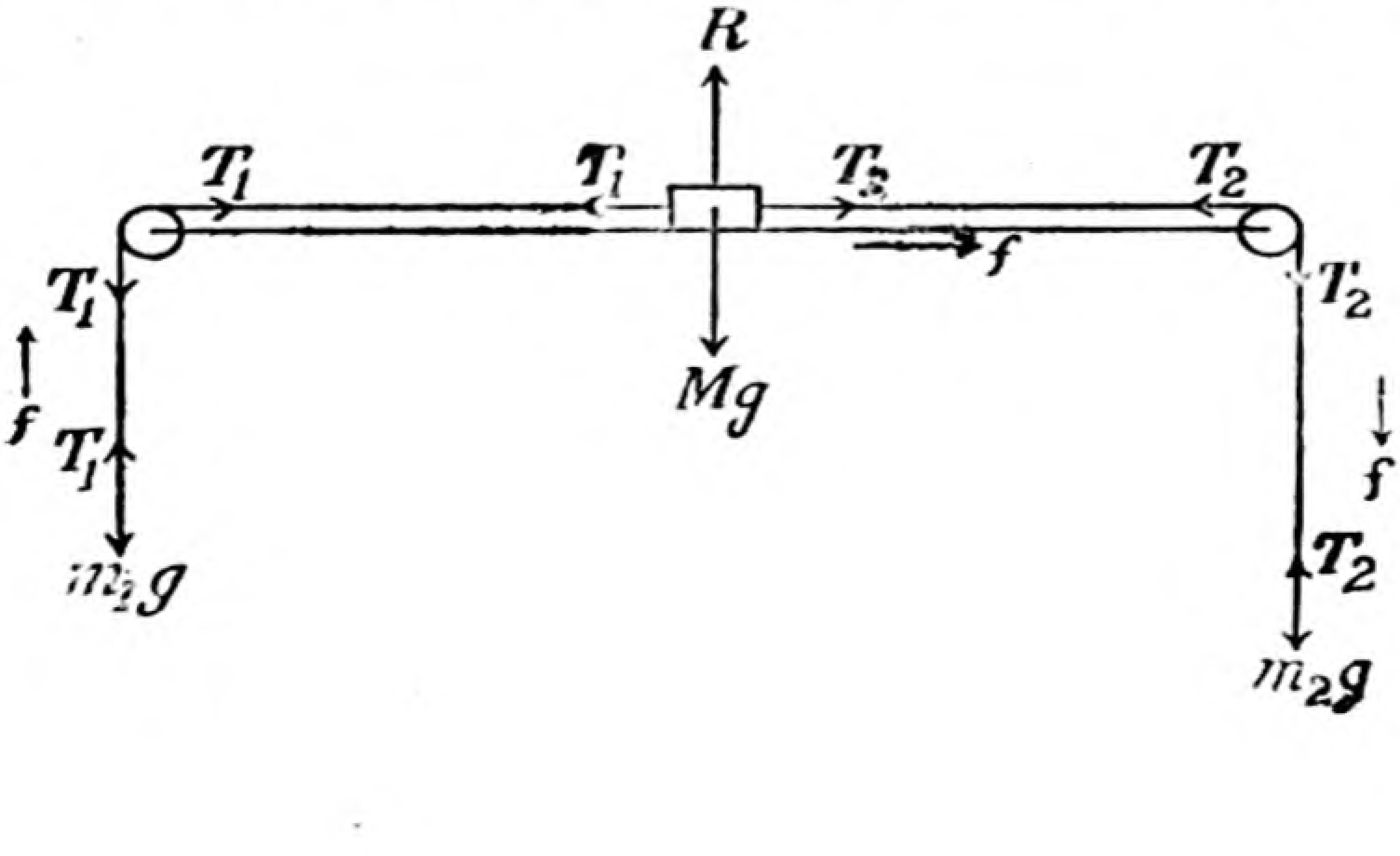
or

$$\frac{4}{M}=\frac{1}{m_1}+\frac{1}{m_2}.$$

Ex. 9. *A string passing across a smooth table at right angles to two opposite edges has attached to it at the ends two masses m_1, m_2 ($m_2>m_1$) which hang vertically. A particle of mass M is also attached to the portion of the string which is on the table. Show that the acceleration of the system when released is*

$$\frac{m_2-m_1}{m_1+m_2+M}g.$$

Find also the tensions of the string on either side of M.



Sol. Let T_1 be the tension in the string connecting m_1 and M and T_2, the tension in the other string.

Since $m_2 > m_1$, mass m_1 moves vertically upwards and mass m_2 moves vertically downwards.

Let f be the common acceleration.

Then for the mass m_1, $m_1f = T_1 - m_1g$ ...(1)

for the mass m_2, $m_2f = m_2g - T_2$ ...(2)

for the mass M, $Mf = T_2 - T_1$ ...(3)

Adding these equations,

$$(m_1+m_2+M)f=(m_2-m)_1g$$

$$\therefore \qquad f=\frac{m_2-m_1}{m_1+m_2+M}g.$$

Substituting the value of f in (1) and (2), we can find the value of T_1 and T_2.

Exercise III (C)

[**Note.** For all practical purposes, rotational inertia and friction between the pulley and its axle may be neglected.]

1. A light string carrying two masses of 14 gm. and 18 gm. hangs over a smooth frictionless pulley. Find the acceleration of the system and the distance moved by the greater mass in 5 seconds.

2. In an Atwood machine a mass of 120 gm. is tied to one end, what mass should be attached to the other end so that the acceleration of the system may be $\frac{1}{4}g$.

3. Two particles, of masses 7 and 9 lb., are connected by a light string passing over a smooth pulley. Find (1) their common acceleration, (2) tension of the string, (3) the velocity at the end of 5 seconds, and (4) the distance described in 5 seconds.

4. The sum of two weights of an Atwood's machine is 16 lb. The heavier weight descends through 64 ft. in 4 sec. What is the value of each weight ? (*P.U.*)

5. Calculate the value of gravity from the following data calculated from the experiment done with the help of an Atwood machine.

Weight on each side = 24 gm.

Weight of rider = 1 gm.

The greater mass fell through 10 cm. in 1 sec.

6. If the string of an Atwood's machine can bear a strain of only $\frac{1}{8}$ of the sum of the two weights, show that the least possible acceleration is $\frac{\sqrt{3}}{2}g$.

7. Two heavy particles are attached together by a string passing over a smooth pulley. If the string can only bear a strain equal to

one quarter of the sum of the two weights ; show that the least possible acceleration is $g/\sqrt{2}$ and that the greater weight cannot be much less than six times the smaller.

8. An Atwood's machine is set up in a lift accelerating upwards with 2 ft./sec^2. If the masses are 3 and 5 oz., calculate the tension and the length of time that elapses before 9 ft. of thread passes over the pulley.

9. Two scale-pans, each of mass m are suspended by a light string over a light pulley and in them are placed masses m_1 and m_2 $(m_1 > m_2)$. Find the reactions of the pans during the motion.

10. Two weights are connected by a string passing over a light frictionless pulley. One is 12 lb. and the other is 11 lb. They are released from rest, and after 2 seconds 2 lb. are removed from the heavier weight. How soon will they be at rest again, and how far will they have moved between the instant of release and that of coming to rest again ?

11. A mass of 5 lb. is placed on a smooth horizontal table 6 ft. high at a distance of 18 ft. from the edge and connected by a light string 18 ft. long to mass of 3 lb. on the edge of the table. If the 3 lb. mass is pushed gently over the edge, find (*i*) how long it takes to reach the ground ; (*ii*) how much longer the 5 lb. mass takes to reach the edge.

12. A mass of 20 lb. on a rough horizontal table is connected by a light string passing over a smooth pulley at the edge of the table and supporting a mass of 4 lb. which hangs freely. The frictional force on the 20 lb. mass is 3 lb. wt. and the 4 lb. mass is originally 16 in. above the floor. The system is released and after one second the string breaks. With what velocity will the 4 lb. mass reach the floor ? How far will the 20 lb. mass move after the string breaks, assuming it does not reach the edge of the table ?

13. Two particles P and Q are attached to the ends of a light string. P is placed on a smooth horizontal table ; while Q hangs over the edge. The system moves freely under gravity. The mass of Q is 1 lb. If P moves 8 ft. in the first second, from rest, find the mass of P. (*P.U.*)

14. Masses m, m' of an Atwood's machine move from rest a distance d ; m being greater than m'. Find the mass that must be removed from m so that the motion may continue a further distance $2d$ in the same sense.

Answers

1. $f = 4$ cm. per sec.^{2} ; S$=50$ cm. **2.** 72 gm.

3. (*i*) $f = 4$ ft./sec^2. (*ii*) $7\frac{7}{8}$ lb. wt. (*iii*) 20 f.p.s. (*iv*) 50 ft.

4. 10 lb. and 6 lb. 5. $g=980$ cm./sec^2.

8. $3\frac{63}{64}$ oz. ; 1·46 seconds.

9. $\frac{2m_1(m+m_2)}{m_1+m_2+2m}\cdot g$; $\frac{2m_2(m+m_1)}{m_1+m_2+2m}g$.

10. $1\frac{19}{23}$ sec. ; $5\frac{171}{529}$ feet.

11. (*i*) 1 sec. (*ii*) 1 sec.

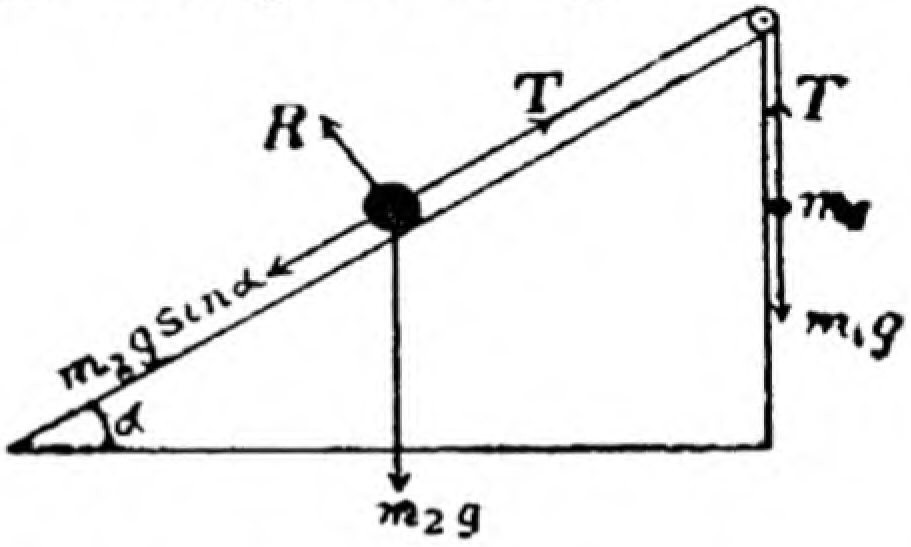
12. $v=6\frac{2}{3}$ f.p.s. ; $2\frac{2}{9}$ inch. 13. $P=1$ lb.

14. $\frac{3(m^2-m'^2)}{3m+m'}$.

(C) MOTION ON AN INCLINED PLANE

3·9 *A mass m_1 hanging vertically pulls a mass m_2 up a smooth inclined plane of inclination α, by means of an inextensible string passing over a smooth pulley at the top of the plane, the part of the string between m_2 and the pulley being parallel to the line of greatest slope ; to find the motion.*

The tension T of the string is same through its length. The velocity and acceleration of m_1 vertically downwards are equal to the velocity and acceleration of m_2 up the plane. Let f be the common acceleration.



The equation of motion for m_1 is

$$m_1f=m_1g-\mathrm{T} \quad \ldots(1)$$

The equation of motion for m_2 up the plane is

$$m_2f=\mathrm{T}-m_2g\sin\alpha \quad \ldots(2)$$

Solving (1) and (2), we have

$$f=\frac{m_1-m_2\sin\alpha}{m_1+m_2}$$

and

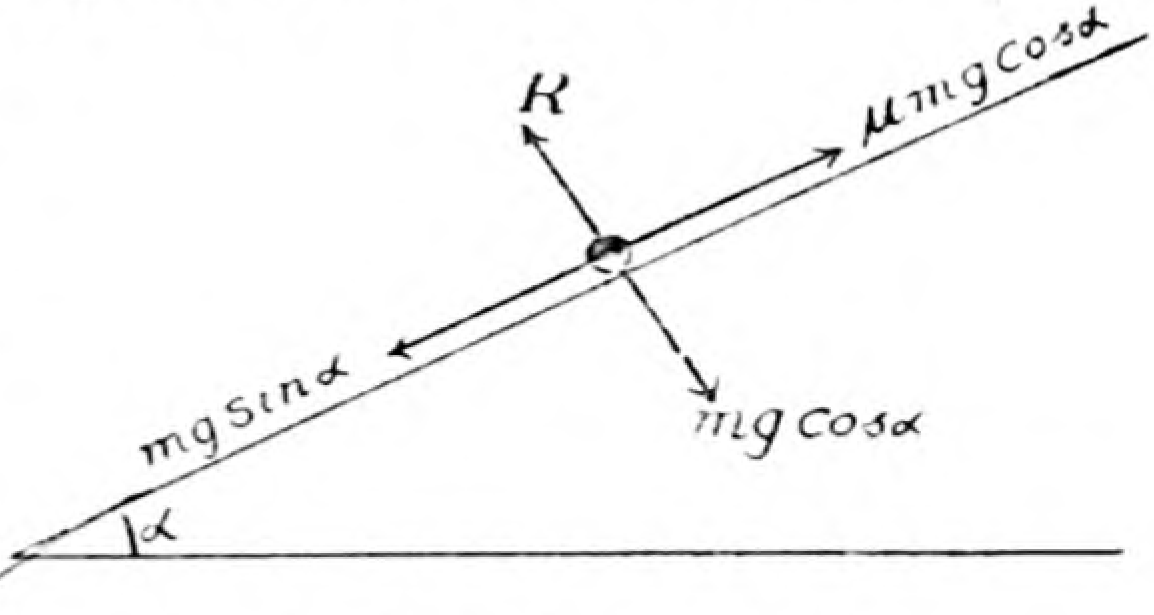
$$\mathrm{T}=\frac{m_1m_2(1+\sin\alpha)}{m_1+m_2}g.$$

If $\alpha=0$, plane is horizontal and we get the results when m_2 is placed on a horizontal plane.

If $\alpha=90°$, we get the results when both the particles are suspended vertically as in Atwood's machine.

3·91 **Rough Inclined Plane** : *A body slides along a rough inclined plane of inclination α, coefficient of friction μ, to find the motion.*

Let the body slide down the plane so that the friction acts up the plane. Let f be the acceleration.



Since there is no motion perpendicular to the plane,

$$R = mg\cos\alpha.$$

The equation of motion down the plane is

$$mf = mg\sin\alpha - \mu R$$
$$= mg\sin\alpha - \mu\, mg\cos\alpha$$

or $$f = g(\sin\alpha - \mu\cos\alpha).$$

If $\sin\alpha < \mu\cos\alpha$, *i.e.* $\tan\alpha < \mu$, the body will not move.

If the body is projected up the plane, the sign of μ will be changed, and the retardation down the plane is

$$g(\sin\alpha + \mu\cos\alpha).$$

3·92 *A mass m_1, hanging vertically, pulls a mass m_2 up a rough inclined plane of inclination α, by means of an inextensible string passing over a pulley at the top of the plane, to find the motion.*

The plane being rough, the force of friction μR acts down the plane as the motion of m_2 is up the plane.

Since m_2 does not move perpendicular to the plane.

$$R = m_2 g\cos\alpha \qquad \text{...(1)}$$

Let f be the common acceleration of the system and T the tension of the string.

The equation of motion for m_1 is

$$m_2 f = m_1 g - T \qquad \text{...(2)}$$

The equation of motion for m_2 is

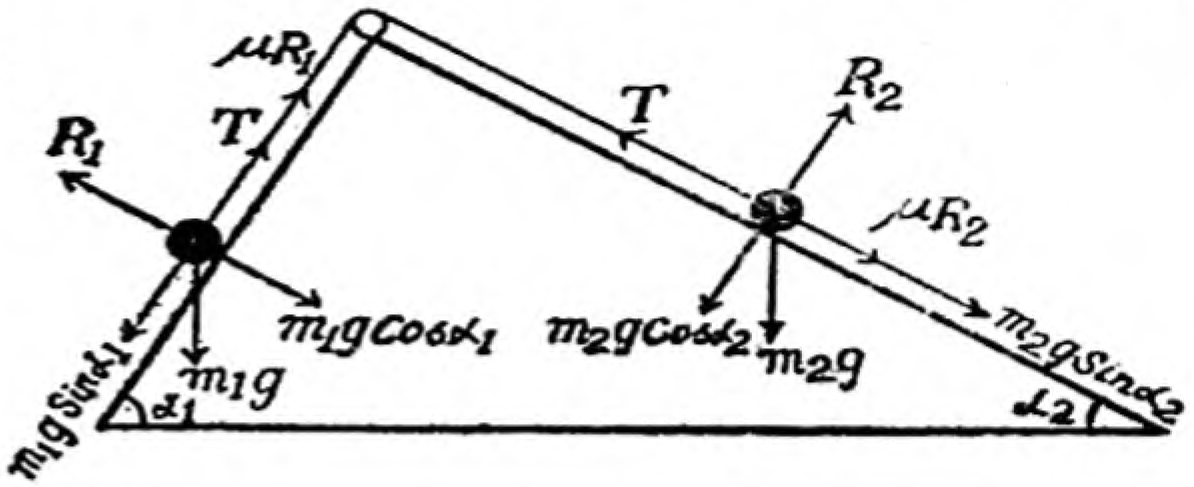
$$m_2 f = T - m_2 g\sin\alpha - \mu R$$
$$= T - m_2 g\sin\alpha - \mu m_2 g\cos\alpha$$
$$= T - m_2 g(\sin\alpha + \mu\cos\alpha) \qquad \text{...(3)}$$

From (2) and (3), we get

$$(m_1 + m_2)f = m_1 g - m_2 g(\sin\alpha + \mu\cos\alpha).$$

Ex. 10. *Two equally rough planes, coefficient of friction μ, of equal height and of inclinations α_1, α_2 are placed back to back, masses m_1 and m_2 are placed on them and are connected by a light inextensible string passing over a pulley at the common vertex of the two planes (the string being parallel to the lines of greatest slopes of the planes through the common vertex). If m_1 descends, find the motion.*

Sol. Let f be the common acceleration and T, the tension of the string.



Since the mass m_1 moves down the plane, its equation of motion is

$$m_1 f = m_1 g \sin \alpha_1 - \mu R_1 - T$$

and because there is no motion of m_1 perpendicular to the plane,

$$R_1 = m_1 g \cos \alpha_1$$

$$\therefore \quad m_1 f = m_1 g \sin \alpha_1 - \mu m_1 g \cos \alpha_1 - T$$

$$= m_1 g(\sin \alpha_1 - \mu \cos \alpha_1) - T \qquad \ldots(i)$$

Since the mass m_2 moves up the plane, its equation of motion is

$$m_2 f = T - m_2 g \sin \alpha_2 - \mu R_2$$

whereas $R_2 = m_2 g \cos \alpha_2$

so that $m_2 f = T - m_2 g(\sin \alpha_2 + \mu \cos \alpha_2) \qquad \ldots(ii)$

Adding (*i*) and (*ii*), we get

$$(m_1 + m_2) f = g[m_1(\sin \alpha_1 - \mu \cos \alpha_1) - m_2(\sin \alpha_2 + \mu \cos \alpha_2)].$$

Exercise III (D)

1. A body slides down a rough incline of slope 30° to the horizon. If the coefficient of friction between the plane and the body be $\frac{\sqrt{3}}{4}$, find the acceleration and the time taken by the body to slide down the plane through 8 ft. from rest. (*P.U.*)

2. A heavy particle of mass 15 lb. resting on a rough inclined plane ($\mu = \frac{1}{2}$) of height 12 ft. and base 16 ft. is connected by a fine string which passes over a smooth pulley on the top of the plane, with two particles of masses 1 lb and 16 lb. which hang vertically. If after the system has been left free to move for one second the particle of mass 16 lb. is taken away, how long after would the system be momentarily at rest? (*P.U.*)

3. A body of mass 10 lb., is placed on a rough plane, whose coefficient of friction is $\frac{1}{\sqrt{3}}$ and whose inclination to the horizon is 30°; if the length of the plane be 4 feet and the body be acted on by a force, parallel to the plane, equal to 15 lb. weight, find the time that elapses before it reaches the top of the plane and its velocity there.

4. A particle slides down a rough inclined plane, whose inclination to the horizon is $\frac{\pi}{4}$ and whose coefficient of friction is $\frac{3}{4}$; show that the time of descending any space is twice what it would be if the plane were perfectly smooth. (*P.U.*)

5. Two rough planes, inclined at 30° and 60° to the horizon and of the same height, are placed back to back ; masses of 5 and 10 lb. are placed on the faces and connected by a string passing over the top of the planes ; if the coefficie friction be $\frac{1}{\sqrt{3}}$, find the resulting acceleration.

6. A car travelling at the rate of 15 m p h. comes to the top of a rough incline of length 440 yds., inclination 30° and coefficient of friction $\frac{\sqrt{3}}{4}$. Find the distance the car will describe on an equally rough horizontal road at the foot of the incline before coming to rest.

7. Two weights m and m' are connected by a string. The weight m hangs vertically and draws m' up a rough plane of inclination α and coefficient of friction μ. Prove that the acceleration is less than it would be if the plane were smooth by an amount

$$\frac{\mu m' \cos \alpha}{m+m'} g.$$

8. A body slides from rest a length l down a rough plane of inclination α and comes to rest after sliding a distance of S along a horizontal plane. If it be assumed that the body experiences no shock in passing from one plane to other, show that

$$S \sin \lambda = l \sin(\alpha - \lambda)$$

where λ is the angle of friction.

Answers

1. 4 ft./sec^2. **2.** $\frac{1}{14}$ sec. **3.** $\sqrt{2}/2$ sec., $8\sqrt{2}$ ft /sec.

5. Larger descends with acc.$=\frac{2\sqrt{3}-3g}{9}$.

6. $\frac{2761\sqrt{3}}{36}$ yards.

Miscellaneous Exercise on Chapter III

1. A train weighing W tons is moving with an acceleration f, when a carriage of weight w tons is suddenly detached from it. Find the change in the acceleration of the train (resistance neglected).

2. A train of mass 120 tons is travelling with a uniform speed the resistance due to friction, etc., being 14 lb. wt. per ton. If a portion of mass 20 tons is slipped, how much will the other portion have gained on it in 12 seconds assuming the pull of the engine and the resistance per ton to be the same as before ?

3. A train of 200 tons, uniformly accelerated, acquires in two minutes from rest a velocity of 30 m.p.h. Show that if the coefficient of friction be 0·18, the part of the load carried by the driving wheels of the engine cannot be less than 12·7 tons.

Show also that if the part of weight of the engine supported by the driving wheels is 25 tons, the velocity at the end of a minute after starting will be less than $29\frac{5}{11}$ miles per hour.

4. A small balloon of total mass 10 lb. at a height of 100 ft. above the ground is moving upwards with a velocity of 60 ft./sec. and acceleration 4 ft./sec.^{2} when a mass of 1 lb. becomes detached. Find the new acceleration of the balloon assuming that the upward force acting upon it remains unaltered.

Also find how many seconds it takes the 1 lb. mass to reach the ground neglecting air resistance.

5. Prove that shortest time from rest to rest in which a steady load of P tons can lift a weight W tons through a vertical distance h feet is

$$\sqrt{\frac{2h}{g}\cdot\frac{P}{p-W}}\ \text{sec.}$$ (*D.U.*)

6. One force acting on a certain mass, originally at rest, causes it to move through a cm. in the pth second of its motion and a second force couses it to move through b cm. in qth second of its motion. Determine (*i*) the measure of the second force, when the first force is taken as the unit and (*ii*) the measure of the first force in dynes when the mass on which it acts contain m grammes. (*P.U.*)

7. Masses m_1 and m_2 in an Atwood's machine are allowed to move from rest any distance x. If m_1 is greater than m_2, show that the mass which must suddenly be removed from m_1 at the end of distance x, so that the motion in the same sense may continue a further distance nx is

$$\frac{(n+1)(m_1^2-m_2^2)}{(n+1)m_1+(n-1)m_2}.$$ (*P.U. 1960*)

8. A man of mass 12 stones and a sack of mass 10 stones are suspended over a smooth pulley by a rope of negligible weight. Find the pressure on the pulley and the acceleration of the man.

If the man pull himself up the rope so as to diminish his acceleration by one-half, find the upward acceleration of the sack and show that the acceleration upwards of the man relative to the rope is

$$\frac{g}{10}.$$ (*P.U.*)

9. Two particles of weights W_1 and W_2 ($W_1 > W_2$) are connected by a light intextensible string, which passes over a small smooth fixed pulley. If the weights start from rest from positions where the vertical distance between them is c, the weight W_1 being at a higher level, find the interval of time t which will elapse before distance between them is d, taking $d > c$. (*P.U. 1961 S*)

10. The two masses in an Atwood's machine are each 240 grammes, and an additional mass of 10 grammes being placed on one of them it is observed to descend through 10 metres in 10 seconds; hence show that $g=980$.

11. A string passing over a smooth pulley carries a mass $4m$ at one end and a pulley of mass m at the other. A string carrying masses m and $2m$ at its end passes over the latter pulley. Find the acceleration of the mass $4m$ when the system is moving freely under gravity.

12. A mass m_1 hanging at the end of a string, draws a mass m_2 along the surface of a smooth table; if the mass on the table be doubled the tension in the string is increased by one-half; find the ratio $m_1 : m_2$.

13. A mass m lying on a smooth horizontal table is attached to a string which after passing over the edge of the table hangs in a loop on which a heavy smooth ring of mass M is threaded and then passes over a smooth fixed pulley and supports a mass m'. If the free portions of the string are vertical and the whole system lies in a vertical plane, determine the tension of the string and show that the mass M will remain at rest provided

$$\frac{2}{M}=\frac{1}{m}+\frac{1}{m'}.$$

14. Two masses m_1, m_2 lie on a smooth horizontal table. They are joined to a string on which slides a smooth ring of mass m_3. If the two portions of the string lie close together in parallel straight lines and the ring hangs over the edge, prove that when released the tension in the thread is

$$\frac{2g}{\dfrac{4}{m_3}+\dfrac{1}{m_2}+\dfrac{1}{m_1}}$$

and that the acceleration of the ring downwards is $\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$ times the tension.

15. A body of weight W_1 hangs vertically from a string which is tied to a body of weight W_2 on a horizontal table. The coefficient of friction between the second body and the table is μ, while owing to the friction at the edge of the table the tension of the vertical part of the string must be n times that of the horizontal part. Find the acceleration and the tensions. Find the relation between W_1, W_2, so that the tension of the horizontal part of the string may be equal to W_2.

16. A particle is projected up a rough plane of inclination d with velocity u and returns to its starting point, after coming to rest with velocity v. Show that the coefficient of friction

$$=\frac{u^2-v^2}{u^2+v^2}\tan\alpha=\frac{t_2{}^2-t_1{}^2}{t_2{}^2+t_1{}^2}\tan\alpha$$

where t_1, t_2 are the times of ascent or descent.

17. A mass m is drawn up a smooth inclined plane, of height h and length l, by means of a string passing over the vertex of the plane, from the other end of which hangs a mass m'. Show that, in order that m may just reach the top of the plane, m' must be detached after m has moved through a distance

$$\frac{m+m'}{m'}\,\frac{hl}{h+l}.$$

18. A particle of mass $2m$ is held in contact with a rough plane of inclination α to the horizontal, and is connected by a light inextensible string passing over a pulley at the top of the plane to a freely hanging particle of mass m. Assuming that angle α is sufficiently large to ensure that the particle $2m$ will move *down* the plane when the system is released, prove that the ensuring acceleration down the plane and the tension in the string will be given by

$$f=(2\sin\alpha-2\mu\cos\alpha-1)\frac{g}{3}$$

$$T=\frac{2mg}{3}(\sin\alpha-\mu\cos\alpha+1).$$

Find also the condition which μ and α must satisfy in order that the particle may move up the plane. *(P.U. 1956)*

Answers

1. $\frac{wf}{W-w}$. **2.** 17·28 ft. **4.** 8 f.p.s. ; 5 sec.

6. (*i*) $\frac{b(2p-1)}{a(2q-1)}$, (*ii*) First force$=\frac{2am}{p-1}$ dynes.

8. $\frac{8}{55}g$. **11.** $\frac{g}{23}$ downwards. **12.** 2 : 1.

15. $\frac{W_1+W_2(1+\mu)g}{W_1+nW_2}$; $\frac{nW_1W_2(1+\mu)g}{W_1+nW_2}$; $\frac{W_1-\mu nW_2}{W_1+nW_2}$; $W_1 : W_2=n : \mu$.

$F = R\mu$

Ac =

Ma

V. Imp

CHAPTER IV

SIMPLE HARMONIC MOTION

(A) Motion in a straight line with variable acceleration

4·1 Equations of Motion : In chapter I, we obtained three different expressions for acceleration, *viz.*,

$$f=\frac{dv}{dt} \text{ or } \frac{d^2s}{dt^2} \text{ or } v\frac{dv}{ds}.$$

Since Impressed Force $P=mf$,

$\therefore$ Equation of motion takes three different forms :

I. $P=m\dfrac{dv}{dt}$ (to be used when P is a function of t or v)

II. $P=m\dfrac{d^2s}{dt^2}$ (,, ,, ,, P ,, ,, t)

III. $P=mv\dfrac{dv}{ds}$ (,, ,, ,, P ,, ,, s or v)

The use of these equations is illustrated in the examples.

Ex. 1. *A particle of mass m begins to move with initial velocity of ka from a distance a, towards a fixed point O in the straight line. If the force per unit mass is k^2s ; show that though it gradually approaches O but would never reach it.* $(k>0)$

Sol. Force on mass $m=mk^2s$, which is a function of s.

$\therefore$ using III of Art. 4·1, $mv\dfrac{dv}{ds}=mk^2s$

$$2vdv=2k^2sds.$$

Integrating, $v^2=k^2s^2+c.$

But initially, when $s=a,\ v=ka$

$\therefore$ $c=0.$

Whence $v=\pm ks.$

Since the mass is moving towards O, v is negative.

$$\therefore \quad v=\frac{ds}{dt}=-ks$$

or

$$\frac{ds}{s}=-kdt.$$

Integrating $\log s=-kt+c'.$

When $t=0,\ s=a,$ hence $c'=\log a.$

$$\therefore \quad \log s-\log a=-kt$$

or
$$\log\frac{s}{a}=-kt$$

or
$$s=ae^{-kt}=\frac{a}{e^{kt}}.$$

As t increases s decreases. But s cannot be zero for any real value of t. This proves the result.

Ex. 2. *A particle moves in a straight line under the retardation μv^2, where μ is constant. If initially it has the velocity u, show that the distance travelled at the end of t seconds is $\frac{1}{u}$ log $(1+\mu ut)$.*

Sol. The equation of motion is
$$\frac{dv}{dt}=-\mu v^2.$$

Separating the variables and integrating,
$$-\int\frac{dv}{v^2}=\mu\int dt+c_1,$$
where c_1 is constant of integration

or
$$\frac{1}{v}=\mu t+c_1.$$

Initially, when $t=0,\ v=u$

$\therefore$
$$\frac{1}{u}=c_1$$

i.e.
$$c_1=\frac{1}{u}$$

$\therefore$
$$\frac{1}{v}=\mu t+\frac{1}{u}$$

which gives
$$v=\frac{u}{1+\mu ut}$$

which we can write as
$$\frac{dx}{dt}=\frac{u}{1+\mu ut}.$$

Separating the variables and integrating
$$\int dx=\int\frac{u}{1+\mu ut}\,dt+c_2$$
where c_2 is the constant of integration.

$\therefore$
$$x=\frac{1}{\mu}\log(1+\mu ut)+c_2.$$

Initially, when $t=0,\ x=0.$

$\therefore \quad 0=c_2, \quad i.e. \quad c_2=0.$

$$\therefore \qquad x=\frac{1}{\mu}\log(1+\mu ut).$$

Ex. 3. *A particle moves in a straight line starting from a distance c to O, centre of attraction towards, which the force per unit mass is* $\frac{\mu}{x^3}$. *Show that the time required to reach the centre is* $\frac{c^2}{\sqrt{\mu}}$.

(*P.U. 1962, 50*)

Sol. At time t, let P be the position of the particle starting from A, where OA$=c$. Let OP$=x$, O being the centre of force.

The equation of motion is

$$m.v\frac{dv}{dx}=-m.\frac{\mu}{x^3}$$

or

$$v\frac{dv}{dx}=-\frac{\mu}{x^3}.$$

Integrating both sides, we have

$$v^2=\frac{\mu}{x^2}+c_1$$

where c_1 is constant of integration.

Initially (at A), when $x=c$, $v=0$.

$$\therefore \qquad 0=\frac{\mu}{c^2}+c_1$$

$$\therefore \qquad c_1=-\frac{\mu}{c^2}$$

$$\therefore \qquad v^2=\frac{\mu}{x^2}-\frac{\mu}{c^2}$$

$$=\mu\left(\frac{1}{x^2}-\frac{1}{c^2}\right)$$

$$=\mu\frac{c^2-x^2}{c^2x^2}$$

$$v=-\frac{\sqrt{\mu}\sqrt{c^2-x^2}}{cx}$$

or

$$\frac{dx}{dt}=-\frac{\sqrt{\mu}\sqrt{c^2-x^2}}{cx}$$

[—ve sign being taken before the radical because the particle moves towards O, the direction in which x decreases.]

Separating the variables and integrating,

$$-\int\frac{cx.dx}{\sqrt{c^2-x^2}}dx=\sqrt{\mu}\int dt+c_2$$

where c_2 is the constant of integration

or
$$c\sqrt{c^2-x^2}=\sqrt{\mu}\ t+c_2.$$

Initially (at A) when $x=c$, $t=0$.

$$\therefore \qquad 0=\sqrt{\mu}.0+c_2.$$

$$\therefore \qquad c_2=0.$$

$$\therefore \qquad c.\sqrt{c^2-x^2}=\sqrt{\mu}\ t$$

or
$$t=\frac{c\sqrt{c^2-x^2}}{\sqrt{\mu}}.$$

Putting $x=0$, the time required to reach the centre $O=\frac{c^2}{\sqrt{\mu}}$.

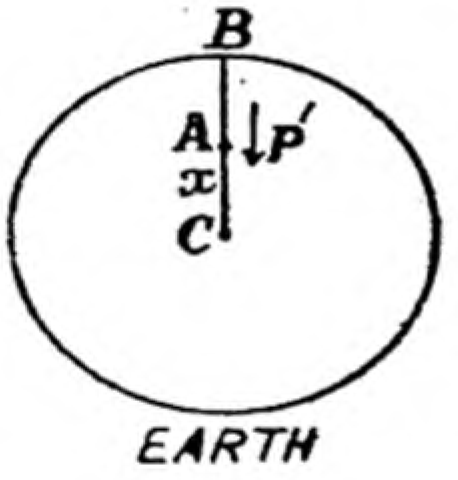
4 2 Motion Under the Attraction of the Earth : Earth attracts a body (mass m) outside its surface with a force varying inversely as the square of its distance from the centre

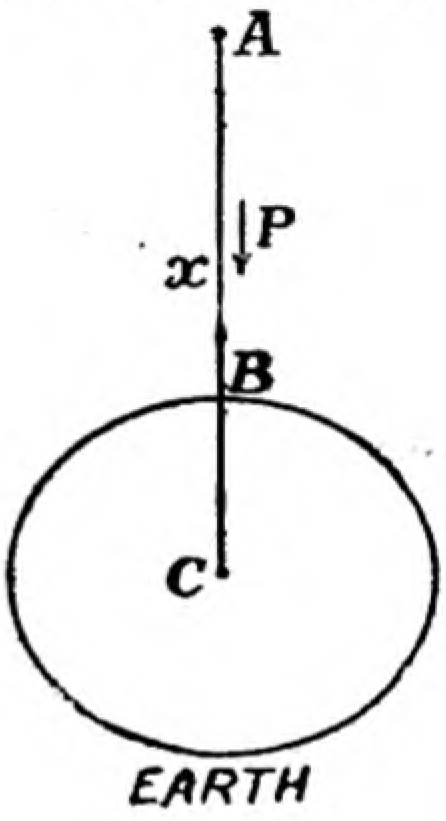
i.e. $$P=-\frac{\mu m}{x^2}, \qquad \ldots(I)$$

while for the motion inside the surface of the earth the force of attraction varies directly as the distance

i.e. $$P'=-\lambda m x \qquad \ldots(II)$$

μ and λ are the constants of variation and may be found as follows.





Let a be the radius of the earth. At B, a point on the surface of the carth, where $x=a$, the attraction is mg ; whence from I

$$-mg=-\frac{\mu m}{a^2}, \text{ which gives } \mu=a^2g.$$

Similarly from II, $-mg=-\lambda ma$, yielding $\lambda=\frac{g}{a}$ ·

$\therefore$ the laws of motion for outside and inside becomes

$$P=-\frac{a^2gm}{x^2} \qquad \text{and} \qquad P'=-\frac{mgx}{a}.$$

The negative sign are in view of the forces being of nature of attraction, which acts in the direction opposite to that in which the distance x increases.

Ex. 4. *Show that the velocity with which a particle must be projected from the surface of the earth so as to escape to infinity is about 7 miles per second.* *(P.U. 1960)*

Sol. For motion outside the surface of earth.

$$P=-\frac{a^2gm}{x^2} \text{ (function of } x\text{)}$$

$$\therefore \qquad mv\frac{dy}{dx}=-\frac{a^2gm}{x^2}$$

or
$$vdx=-a^2g\,\frac{dx}{x^2}$$

Integrating
$$\frac{v^2}{2}=\frac{a^2y}{x}+c \qquad \text{...(I)}$$

Let V be the velocity of projection so that when $x=a$, $v=V$.

$$\therefore \qquad \frac{V^2}{2}=ag+c$$

or
$$c=\frac{V^2}{2}-ag \qquad \text{...(II)}$$

From (I) and (II) $v^2=V^2+\frac{2a^2g}{x}-2ag.$

The equation requires that $v \to 0$ as $x \to \infty$, so that

$$V^2=2ag$$

or
$$V=\sqrt{2ag}=\sqrt{2\times4000\times1760\times3\times32}\ f.p.s.=7\ m\ p.s. \text{ nearly.}$$

Ex. 5. *Show that the time of descent to the centre of force, the force varying inversely as the square of the distance from the centre, through the first half of its initial distance is to that through the latter half as* $\pi+2 : \pi-2$.

Sol. Suppose the particle starts from rest from a distance a.

By the Hypothesis, $v\frac{dv}{dx}=-\frac{\mu}{x^2}$ ...(I)

which gives on integration, $v^2=\frac{2\mu}{x}+c.$

When $x=a$, $v=0$, hence $v^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right)$ ...(II)

$$v=\pm\sqrt{2\mu\left(\frac{1}{x}-\frac{1}{a}\right)}.$$

Since the motion is in the direction opposite to that in which x increases, v is negative.

$$\therefore \qquad \frac{dx}{dt}=v=-\sqrt{\frac{2\mu}{a}}\sqrt{\frac{a-x}{x}}$$

or
$$\sqrt{\frac{2\mu}{a}}\,dt=-\sqrt{\frac{x}{x-a}}dx.$$

Let t_1 and t_2 denote the times for the two halves.

$$\therefore \quad \sqrt{\frac{2\mu}{a}}\, t_1 = -\int_a^{a/2} \sqrt{\frac{x}{a-x}}\, dx = \int_{a/2}^{a} \sqrt{\frac{x}{a-x}}\, dx$$

$$= \int_{\pi/4}^{\pi/2} 2a \sin^2 \theta\, d\theta = \frac{a}{4}(\pi+2)$$

(putting $x = a \sin^2 \theta$)

Similarly $\sqrt{\frac{2\mu}{a}}\, t_2 = -\int_{a/2}^{0} \sqrt{\frac{x}{x-a}}\, dx = \frac{a}{4}\ (x-2).$

Hence $t_1 : t_2 = \pi+2 : \pi-2.$

Exercise IV (A)

1. If the time taken by a particle to move a distance s in a st. line is given by $t = as^2 + bs$; show that it moves under a retardation varying as the cube of the velocity.

2. A particle initially at rest, moves from a fixed point in a st. line so that at the end of t seconds, its acceleration is $\sin t + \frac{1}{(t+1)^2}$. Show that its distance from the fixed point at the end of π seconds is $2\pi - \log(\pi+1)$.

3. Prove that it is impossible for a particle to move from rest so that its velocity varies as the distance described from the commencement of the motion.

If the velocity varies as (distance)^n, show that n cannot be greater than $\frac{1}{2}$.

4. A particle moving in a st. line is subject to a resistance which produces a retardation Kv^3, where v is the velocity and K is a constant. Show that v and t are given in terms of x by the equations

$$v = \frac{u}{1+Kxu}\ ; \quad t = \frac{1}{2}Kx^2 + \frac{x}{u},$$

where u is the velocity at $t=0$. (*P.U. 1960*)

5. A particle moves in a st. line from a fixed point O with its velocity V under a force which produces an acceleration μx, where x is the distance from O. Find the time taken in moving a distance x and the distance moved in time t. Find also the time taken for the velocity to be increased to 2V. (*P.U.*)

6. A particle moves towards a centre of attraction, starting from rest at a distance a from the centre. If the velocity at a distance x from the centre is $\mu\sqrt{\frac{a^2-x^2}{x^2}}$; find the law of the force and the time it takes to arrive at the centre. (*P.U. 1955*)

7. A particle is attracted by a force to a fixed point varying inversely as the (distance)^n. If the velocity acquired in falling from the infinite distance to a distance a from the centre be equal to the

relocity acquired in falling from rest at a distance a to a distance $\frac{a}{4}$, then show that $n=\frac{3}{2}$.

8 A particle starts from rest and its acceleration at any time t is $f-kt^2$, where f and k are constants. Prove that the maximum velocity u of the particle is $\frac{2}{3}\sqrt{f^3/k}$ and that the space described by it before it acquire this velocity is $15u^2/16f$.

9. A particle moving in a st. line has an acceleration $\mu\left(x+\frac{a^4}{x^3}\right)$ towards a fixed point in the line at a distance x. If it starts from rest at a distance a from the fixed point, show that it arrives at it in time $\frac{\pi}{4\sqrt{\mu}}$.

(P.U.)

10. (*a*) A particle is let fall from rest from a point outside the earth at a distance b from its centre. Prove that the square of its velocity on reaching the centre is $ga\left(3-\frac{2a}{b}\right)$; where a is the radius of the earth and g is the value of gravity at the surface of the earth.

(P.U. 1959 S)

(*b*) Show also that the velocity on reaching the surface of the earth is

$$\sqrt{2ga\left(1-\frac{a}{b}\right)}.$$

(P.U. 1960)

Answers

5. $t=\frac{1}{\sqrt{\mu}}\sinh^{-1}\frac{\sqrt{\mu}x}{V}$; $x=\frac{V}{\sqrt{\mu}}\sinh\sqrt{\mu}\,t$

$T=\frac{1}{\sqrt{\mu}}\log(2+\sqrt{3})$.

6. Law of force is $\frac{-m\mu^2a^2}{x^3}$; $T=\frac{a}{\sqrt{\mu}}$.

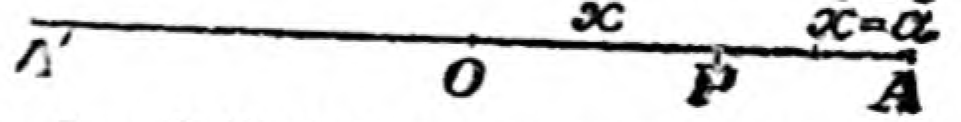
(B) Simple Harmonic Motion

4·3 Simple Harmonic Motion or S.H.M. is one of the outstanding examples of motion with a variable acceleration.

4·31 Def. *When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point, its motion is called* **Simple Harmonic Motion** *or* **S.H M.**

The fixed point is called the *mean position* or the *centre.*

Case 1. Let O be the fixed point on the line and let the particle start from rest from the point A where OA$=a$.



Let P be the position of the particle at time t, where OP$=x$. Then, by definition, magnitude of the acceleration is μx and since the acceleration is directed towards O, therefore, the acceleration in the positive direction of x-axis is $-\mu x$.

$\therefore$ the equation of the simple harmonic motion is

$$\frac{d^2x}{dt^2}=-\mu x \quad \text{or} \quad v\frac{dv}{dx}=-\mu x$$

or $$2v\,dv=-2\mu x\,dx \qquad \text{...(I)}$$

Integrating, $$\frac{v^2}{2}=-\mu\frac{x^2}{2}+C$$

where C is a constant of integration.

Initially when $x=a,\ v=0.$

$$\therefore \quad 0=-\mu\frac{a^2}{2}+C$$

$$\therefore \quad C=\tfrac{1}{2}\mu a^2$$

$$\therefore \quad \frac{v^2}{2}=-\mu\frac{x^2}{2}+\frac{1}{2}\mu a^2$$

or $$v^2=\mu(a^2-x^2) \qquad \text{...(II)}$$

$$\therefore \quad v=\pm\sqrt{\mu}\sqrt{a^2-x^2} \qquad \text{...(II)}$$

or $$\frac{dx}{dt}=\pm\sqrt{\mu}\sqrt{a^2-x^2}$$

This gives the velocity v for any displacement x.

Since the particle is moving towards O, x decreases as increases so that $\frac{dx}{dt}$ is *negative.*

$$\therefore \quad \frac{dx}{dt}=-\sqrt{\mu}\sqrt{a^2-x^2}$$

or $$\frac{dx}{\sqrt{a^2-x^2}}=\sqrt{\mu}\,dt.$$

Integrating,

$$-\int\frac{dx}{\sqrt{a^2-x^2}}=\int\sqrt{\mu}\,dt+C_1$$

or $$\cos^{-1}\frac{x}{a}=\sqrt{\mu}\,t+C_1$$

where C_1 is a constant to be determined.

Initially at A, $t=0$ when $x=a$.

$$\therefore \quad C_1=0$$

$$\therefore \quad \cos^{-1}\frac{x}{a}=\sqrt{\mu}\,t$$

or $$x=a\cos\sqrt{\mu}\,t \qquad \text{...(III)}$$

Case 2. *If the particle starts from O towards A, v is positive. In this case $x=0$ when $t=0$.*

$$\therefore \qquad \frac{dx}{dt}=\sqrt{\mu}\,\sqrt{a^2-x^2}$$

or
$$\frac{dx}{\sqrt{a^2-x^2}}=\sqrt{\mu}\,dt.$$

Integrating,

$$\sin^{-1}\frac{x}{a}=\sqrt{\mu}\,t+C_2$$

where C_2 is a constant.

Now $t=0$ when $x=0$.

$$\therefore \qquad C_2=0$$

$$\sin^{-1}\frac{x}{a}=\sqrt{\mu}\,t$$

$$\therefore \qquad x=a\sin\sqrt{\mu}\,t \qquad \text{...(IV)}$$

Thus $\quad x=a\cos\sqrt{\mu}\,t$, *from (III)*

or $\quad x=a\sin\sqrt{\mu}\,t$, *from (IV)*

according as t is measured from A or O.

4·32 Nature of Motion : The velocity v for any displacement x is given by

$$v=\pm\sqrt{\mu}\,\sqrt{a^2-x^2} \qquad \text{...(II)}$$

When $\quad x=a, \quad v=0.$

Also when $\quad x=-a, u=0.$

$\therefore$ if A′ is a point on the other side of O such that $OA'=a$, he particle comes to rest at A′.

The velocity is maximum when $x=0$.

$\therefore$ at O, $v=\pm\sqrt{\mu}\;a.$ ...(V)

The acceleration is maximum when $x=\pm a$, *i.e.* at the extreme positions so that

$$\text{Maximum Acceleration}=\mu a. \qquad \text{...(VI)}$$

The particle starts from the point A under an accelaration towords O. The velocity, therefore, goes on increasing and the acceleration is decreasing. At O, the mean position, the velocity is maximum (magnitude$=\sqrt{\mu}\,a$) and acceleration is zero. The particle now moves to the left side of O. The acceleration being towards O, velocity will go on decreasing now so that at A′ the velocity vanishes and the particle is instantaneously at rest. At this position, the acceleration is maximum and is towards O. The path is retraced. The motion is, therefore, from A to A′ and back to A and so on.

The motion of the particle is *oscillatory.*

The distance a, the distance of the extreme position (the position of rest) from the centre is called the *Amplitude.*

Velocity will be half the maximum, *i.e.* $\frac{a}{2}\sqrt{\mu}$ at a distance y so that from II, $\frac{a}{2}\sqrt{\mu}=\sqrt{\mu}\sqrt{a^2-y^2}$ or $y=\frac{\sqrt{3}}{2}a.$

4·4 Periodic Motion : Def. A particle is said to have periodic motion when it moves in such a way that it occupies the same position and moves in the same direction with the same velocity after some fixed interval of time. This fixed interval is called the *periodic time.*

4·41 S.H.M. is a Periodic Motion : We shall show now that the simple harmonic motion is periodic whose period is independent of amplitude.

The displacement x of the particle after time t is given by

$$\begin{aligned} x &= a\cos\sqrt{\mu}\,t \\ &= a\cos(\sqrt{\mu}\,t+2\pi) \\ &= a\cos\sqrt{\mu}\left(t+\frac{2\pi}{\sqrt{\mu}}\right) \qquad \text{.. (A)} \end{aligned}$$

Also the velocity of the particle at time t is given by

$$\begin{aligned} v=\frac{dx}{dt} &= -a\sqrt{\mu}\sin\sqrt{\mu}\,t \\ &= -a\sqrt{\mu}\sin(\sqrt{\mu}\,t+2\pi) \\ &= -a\sqrt{\mu}\sin\sqrt{\mu}\left(t+\frac{2\pi}{\sqrt{\mu}}\right) \qquad \text{...(B)} \end{aligned}$$

It follows from (A) and (B) that the particle has the same position, velocity and direction after times $t+\frac{2\pi}{\sqrt{\mu}}$, $t+\frac{4\pi}{\sqrt{\mu}}$, ...*as it had at the time t.*

Hence S.H.M. is a Periodic Motion of period $\frac{2\pi}{\sqrt{\mu}}$.

4·42 *Period for one oscillation or Periodic Time is independent of amplitude.*

Since in S.H.M. the particle has a periodic motion, the period of motion being $\frac{2\pi}{\sqrt{\mu}}$ which is also the time from A to A′ and back to A, *i.e.*, for one oscillation,

$$T=\frac{2\pi}{\sqrt{\mu}}$$

which is independent of a (amplitude).

The period for one vibration is $\dfrac{\pi}{\sqrt{\mu}}$.

44·3 Frequency is the number of complete oscillations in one second. If n is the frequency and T the periodic time, then

$$nT=1$$

or
$$n=\frac{1}{T}=\frac{\sqrt{\mu}}{2\pi}. \qquad \text{...(VIII)}$$

Ex. 7. *A particle moves in a straight line with S.H.M. of period 2 sec. If it starts from rest at a distance of 13 cm. from the centre of the path, show that the greatest velocity and the velocity acquired by it when it has just described 8 cm. are respectively 13π and 12π cm/sec.* (*P.U.*)

Sol. Let μ be the constant of S.H.M.

We have
$$T=\frac{2\pi}{\sqrt{\mu}}=2$$

$\therefore$
$$\mu=\pi^2.$$

Also
$$a=13 \text{ cm.}$$

$\therefore$ greatest velocity
$$=\sqrt{\mu}\, a$$
$$=13\pi \text{ cm./sec.}$$

When the particle has described 8 cm., it is at a distance of 5 cm. from the centre.

$\therefore$
$$x=5 \text{ cm.}$$

Now
$$v=\sqrt{\mu}\,\sqrt{a^2-x^2}$$
$$=\pi\sqrt{169-25}$$
$$=12\pi \text{ cm./sec.}$$

Ex. 8. *Show that a particle executing simple harmonic motion requires one-sixth of the period to move from the position of maximum displacement to one in which the displacement is half the amplitude.*

Sol. If t be measured from the instant when the particle is at rest, then

$$x=a\cos\sqrt{\mu}\, t.$$

Here
$$x=\frac{a}{2}$$

$\therefore$
$$\frac{a}{2}=a\cos\sqrt{\mu}\, t$$

or
$$\cos\sqrt{\mu}\, t=\tfrac{1}{2}$$

$\therefore$
$$\sqrt{\mu}\, t=\frac{\pi}{3}$$

or
$$t=\frac{\pi}{3\sqrt{\mu}}. \qquad \text{...(1)}$$

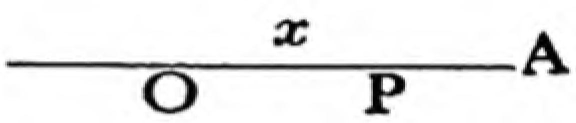
Now the period is given by

$$T=\frac{2\pi}{\sqrt{\mu}} \qquad \ldots(2)$$

whence Imp $$t=\tfrac{1}{6}T.$$

Ex. 9. *A point is performing simple harmonic motion of period T about a centre O, and it passes through the position $P(OP=b)$ with velocity v in the direction OP. Prove that the time that elapses before it returns to P is* $\frac{T}{\pi}\tan^{-1}\frac{vT}{2\pi b}$.

Sol. Let t be the time from A to P. Then



$$x=a\cos\sqrt{\mu}\,t$$

or $$b=a\cos\sqrt{\mu}\,t \qquad =(1)$$

Also $$v=\sqrt{\mu}\sqrt{a^2-b^2}$$

and $$T=\frac{2\pi}{\sqrt{\mu}}$$

which gives $$\sqrt{\mu}=\frac{2\pi}{T}.$$

From (1), $$\cos\sqrt{\mu}\,t=\frac{b}{a}$$

$$\therefore \quad \tan\sqrt{\mu}\,t=\frac{\sqrt{a^2-b^2}}{b}=\frac{vT}{2\pi b}$$

$$\therefore \quad t=\frac{1}{\sqrt{\mu}}\tan^{-1}\frac{vT}{2\pi b}$$

$$=\frac{T}{2\pi}\tan^{-1}\frac{vT}{2\pi b}.$$

$\therefore$ required time $$=2t$$

$$=\frac{T}{\pi}\tan^{-1}\frac{vT}{2\pi b}.$$

Ex. 10. *A particle is moving with S.H.M, and while making an excursion from one position of rest to the other, its distances from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3; prove that the time of complete oscillation is*

$$\frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_2}{2x_2}\right)}.$$

Sol. Since the times are measured from the position of rest, we have for three consecutive seconds t, $t+1$, $t+2$,

$$x_1=a\cos\sqrt{\mu}\,t \qquad \ldots(1)$$

$$x_2=a\cos\sqrt{\mu}\,(t+1) \qquad \ldots(2)$$

$$x_3=a\cos\sqrt{\mu}\,(t+2) \qquad \ldots(3)$$

Adding (1) and (3),

$$\begin{aligned} x_1+x_3 &= a[\cos\sqrt{\mu}\,t+\cos\sqrt{\mu}\,(t+2)] \\ &= a\,.\,2\cos\sqrt{\mu}\,(t+1)\,.\,\cos\sqrt{\mu} \\ &= 2x_2\,.\,\cos\sqrt{\mu}, \qquad \text{[from (2)]} \end{aligned}$$

$$\therefore \qquad \cos\sqrt{\mu}=\frac{x_1+x_3}{2x_2}$$

$$\therefore \qquad \sqrt{\mu}=\cos^{-1}\frac{x_1+x_2}{2x_2}$$

$$\therefore \qquad \text{time period}=\frac{2\pi}{\sqrt{\mu}}$$

$$=\frac{2\pi}{\cos^{-1}\left(\dfrac{x_1+x_2}{2x_2}\right)}.$$

Ex. 11. *A particle of mass 4 grammes vibrates through one centimetre on each side of the middle point of its path making 330 complete vibrations per minute. Assuming its motion to be S.H.M., show that the maximum force upon the particle is 484π^2 dynes.*

(P.U. Pre-Engg. 1967)

Sol. a, the amplitude$=1$ cm.

m, the mass of the particle$=4$ grammes.

Number of revolutions per second$=\frac{330}{60}=\frac{11}{2}$.

Now number of revolutions per second represents frequency.

$$\therefore \qquad \frac{\sqrt{\mu}}{2\pi}=\frac{11}{2}$$

$$\therefore \qquad \sqrt{\mu}=11\pi$$

or

$$\mu=121\pi^2$$

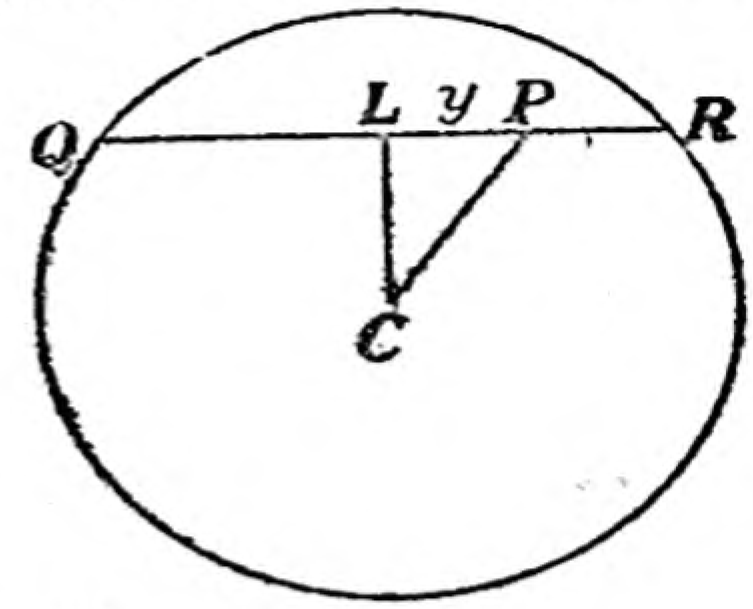
Maximum acceleration

$$\begin{aligned} &=\mu a \\ &=121\pi^2\times 1 \text{ cm./sec.}^2 \\ &=121\pi^2 \text{ cm./sec.}^2 \end{aligned}$$

$$\begin{aligned} \therefore \qquad \text{Maximum force} &=\text{Mass}\times\text{Maximum acceleration} \\ &=4\times 121\pi^2 \text{ dynes} \\ &=484\pi^2 \text{ dynes.} \end{aligned}$$

Ex. 12. *If a smooth straight tunnel be cut between any two points on the earth's surface, find approximately the time in which it would be traversed by a particle starting from rest under the action of gravity.*

Sol. Let C be the centre of the earth, a its radius and QR the tunnel.



Let CL ⊥ QR.

Let P be the position of the particle at a time t, where LP $= y$.

Now the law of force of attraction inside the earth is

$$-\frac{mgx}{a}, \quad i.e. \quad -\frac{mg}{a} . \text{CP}.$$

∴ Component of attraction along

$$\text{LP} = -\frac{mg}{a}\text{CP} . \cos \angle\text{CPL} = -\frac{mg}{a} . y$$

∴ Equation of motion of P is $m\dfrac{d^2y}{dt^2} = -\dfrac{mg}{a}y$

or
$$\frac{d^2y}{dt^2} = -\frac{g}{a}y$$

which represents a S.H.M. whose period

$$= \frac{2\pi}{\sqrt{\dfrac{g}{a}}} = 2\pi\sqrt{\frac{a}{g}}.$$

∴ required time from R to Q

$$= \pi\sqrt{\frac{a}{g}}$$

$$= 3 \cdot 1416\sqrt{\frac{4000 \times 1760 \times 3}{32}}$$

$$= 42 \cdot 5 \text{ sec. nearly.}$$

Exercise IV (B)

1. Define 'Simple Harmonic Motion' and prove that it is periodic. (*P.U. 1966*)

2. A particle moves in a st. line with a force varying directly as its distance from a fixed point on the line. If the force be attraction show that the periodic time is independent of the amplitude. (*P.U.*)

3. A point, moving with S.H.M., has a velocity of 4 ft./sec. when passing through the centre of its path, and its period is π sec. What is its velocity when it has described one foot from the position in which its velocity is zero? Also find the time taken.

4. A point executing S.H.M. passes through two points A and B, 24 inches apart, with the same velocity having occupied 4 sec. in passing from A to B. After another 4 sec. it returns to B. Find the period and the amplitude.

5. If the displacement, velocity and acceleration at a particular instant of a point moving under S.H.M. are 5 cm., 5 cm./sec. and 5 cm./sec^2 ; find the maximum velocity and period of motion.

6. A particle moves in a st. line with S.H.M. of periodic time of 2 sec. If it starts from rest at a distance of 13 cm. from centre of its path, show that the greatest velocity and the velocity acquired when it has just described 8 cm. are 13π and 12π cm. per second respectively. (P.U.)

7. (*i*) A point P moves in a straight line with S.H.M. the centre of motion being O and the extreme position being A. Q is a point in OA such that the time from A to Q is the same as the time from Q to O. Show that $OA^2=2OQ^2$.

(*ii*) If $OA^2=2OQ^2$, show that time from A to Q is the same as the time from Q to O.

8. At the ends of three successive seconds, the distances of a point moving with S.H.M. from its mean position, measured in the same direction are 1, 5, 5. Show that the period of complete oscillation is $\frac{2\pi}{\theta}$ seconds, where $\cos\theta=\frac{3}{5}$. (P.U.)

9. A particle is executing S.H.M. ; A and B are the two points at which its velocity is zero. It passes through a certain point P at intervals of 0 5 and 1·5 seconds with a speed of 3 ft./sec. Determine the maximum speed and also the ratio $\frac{AP}{PB}$. (P.U.)

10. A particle moving in a straight line OAB with S.H.M. has its velocity zero when it is at the points A and B and it has a velocity v when it is half between them. If the distances of A and B with O as origin are a and b respectively, show that the periodic time is $\frac{\pi\, b-a)}{v}$.

11. A point executes S.H.M. such that in two of its positions the velocities are u, v and the two corresponding accelerations α, β ; show that the distance between the two positions is $\frac{v^2-u^2}{\alpha+\beta}$.

Also show that the period of motion is $2\pi\sqrt{\frac{u^2-v^2}{\beta^2-\alpha^2}}$, (P.U.)

And that the amplitude of the motion is

$$\frac{[(v^2-u^2)(\alpha^2v^2-\beta^2u^2)]^{\frac{1}{2}}}{\beta^2-\alpha^2},\ (\beta>\alpha).$$ (P.U.)

12. A horizontal shelf is moved up and down with S.H.M. of period $\frac{1}{2}$ second. What is the greatest amplitude admissible in order that a weight placed on the shelf may not be thrown off.

13. If the displacement of a moving point at any time is given by an equation of the form

$$x = a \cos kt + b \sin kt$$

show that the point executes simple harmonic motion.

If $a=3$, $b=4$, $k=2$, find the period, amplitude, maximum velocity and maximum acceleration of the motion.

14. The speed v of the particle moving along the axis of x is given by the relation $v^2 = n(8bx - x^2 - 12b^2)$ Show that the motion is simple harmonic with its centre at $x=4b$ and amplitude $2b$. Find the time from $x=5b$ to $x=6b$. *(P.U.)*

15. A point moving in a straight line with S.M.H. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the time of complete oscillation is

$$2\pi \sqrt{\frac{x_1^2 - x_1^2}{v_2^2 - v_1^2}}. \qquad \textit{(P.U. 1959, 63)}$$

16. A particle moves with S.H.M. in a straight line. In the first second after starting from rest it travels a distance a and in the next second it travels a distance b in the same direction. Prove that the amplitude of the motion is

$$\frac{2a^2}{3a-b}. \qquad \textit{(P.U. T.D.C. II 1964)}$$

17. A particle is attached to one end of an inextensible string and the other end moves with S.H.M. in a vertical line with n complete oscillations per second. Show that the string will not remain tight during the motion unless $x^2 < \frac{g}{4\pi^2 a}$, where a is the amplitude of motion.

Answers

3. $v=3{\cdot}46$ f.p.s. ; $t=\pi/6$ sec. **4.** 16 sec. ; 16·97 in.

5. Max. vel.$=5\sqrt{2}$ cm. p.s. ; $T=2\pi$ sec.

9. $3\sqrt{2}$ f.p.s. ; $\frac{AP}{BP}=0{\cdot}17$. **12.** $\frac{g}{16\pi^2}$.

13. 3·14 sec. ; 5 ft. ; 10 f.p.s. **14.** $\frac{\pi}{3n}$.

(C) Motion of a Particle attached to Light Elastic String

4·6 Hooke's Law. It states that the tension of an elastic string or spring is proportional to extension.

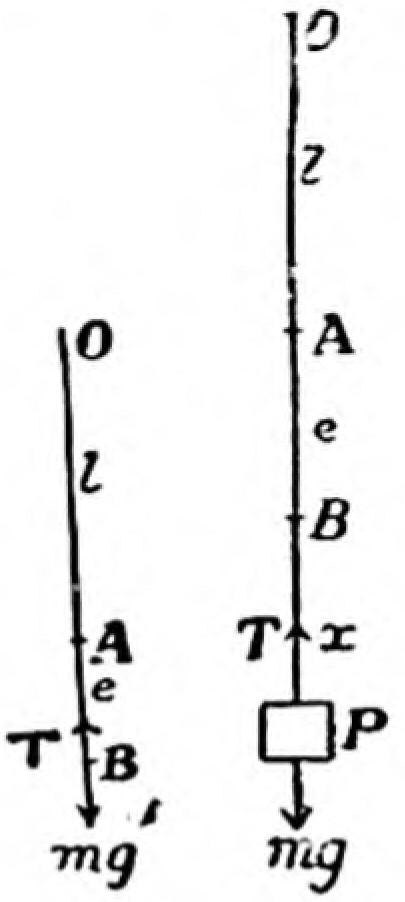
Let T be the tension in an elastic string whose natural length is l. Let $l+x$ be the stretched length. Then x being the extension,

$$T = \lambda \frac{x}{l},$$

where λ is a constant (called the *Modulus of Elasticity*) which depends upon the material and thickness of the string.

4 61 Vertical Elastic Strings. *A particle of mass m hangs from a fixed point by an all elastic string of natural length l and modulus of elastricity* λ, *stretching it, when in equilibrium to a length* $l+e$. *The mass is then pulled down a further distance* $c<b$ *and then let go ; to discuss the ensuing Motion.*

(P.U. 1958 S)



Let a particle of mass m be suspended at one end A of the string the other end O of which is fixed. Let $OA=l$.

Let OB be the length of the string when the mass m hangs in equilibrium and let $AB=l$.

Since $mg=T$

By Hooke's Law,

$$mg=T=\lambda\,\frac{e}{l} \qquad \ldots(1)$$

Let the particle be displaced vertically below B through a distance c. Let P be the position of the particle at time t, where $PB=x$. If T be the tension in the string in this position, we have

$$T=\lambda.\frac{AP}{l}=\frac{\lambda}{l}(e+x)$$

$$=\frac{\lambda e}{l}+\frac{\lambda x}{l}$$

$$=mg+\frac{\lambda x}{l}, \qquad \text{from (1)}$$

$\therefore$ The equation of motion of the particle in the direction BP is

$$m\frac{d^2c}{dt^2}=mg-T$$

$$=mg-\left(mg+\frac{\lambda x}{l}\right)$$

$$=-\frac{\lambda x}{l}$$

or
$$\frac{d^2y}{dt^2}=-\frac{\lambda}{lm}x$$

which is the equation of S.H.M. with period

$$=\frac{2\pi}{\sqrt{\frac{\lambda}{lm}}}=2\pi\sqrt{\frac{lm}{\lambda}}=2\pi\sqrt{\frac{e}{g}}.$$

Hence the particle moves with S.H.M. with its centre of oscillation at B, the equilibrium position. The mass attains maximum velocity at B.

Maximum velocity$=\sqrt{\mu}$ (amplitude)

$$=\sqrt{\frac{\lambda}{lm}}.c=\sqrt{\frac{g}{c}}.c$$ *(P.U. 1961)*

4·62 **The following points may be noted :**

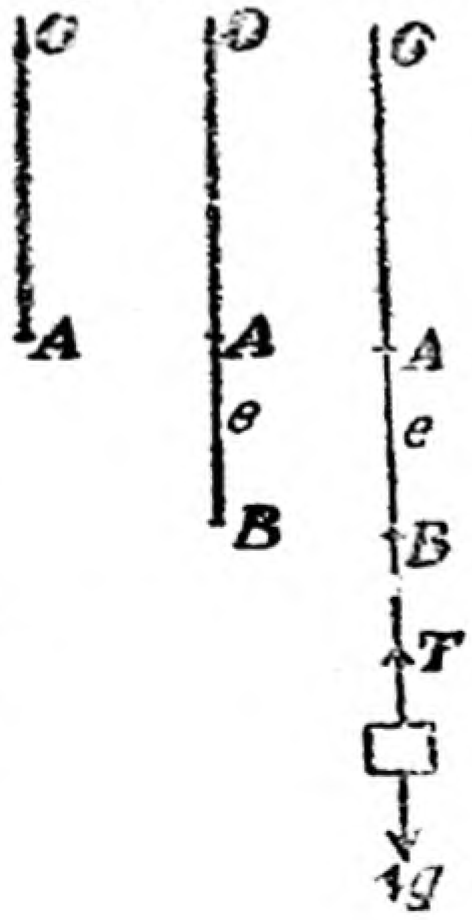
1. The motion is simple harmonic about the equilibrium position B, and not about A.

2. The motion will be simple harmonic as long as the particle is below A, *i.e.* as long as the string is stretched.

3. If the particle rises above A, the motion above A is a free vertical motion under gravity. This case will arise if the particle is pulled below B a distance greater than AB, *i.e.* when $c>e$.

Ex. 13. *A light elastic string whose natural length is 4 ft. is such that a force of 12 lb. wt. will stretch it to twice its natural length. One end of the string is fixed and a mass of 4 lb. is attached to the other. Initially the string is vertical and the mass is held so that the string is just unstretched. The mass is then let go ; prove that the subsequent motion is simple harmonic and find the periodic time.*

Sol. Since the force of 12 lb wt. extends the string to twice its natural length, by Hooke's Law.



$$12g=\lambda.\tfrac{4}{4}$$

$$\therefore \qquad \lambda=12g.$$

Now when the mass of 4 lb. is suspended, let e be the extension in the equilibrium position.

$$\therefore \qquad 4g=\lambda.\frac{e}{4}=12g.\frac{e}{4}$$

$$\therefore \qquad e=\tfrac{4}{3}.$$

Let, at any time t, P be the position of the particle, where BP$=x$. Let T be the tension in the string.

Then $$T=\lambda\,.\,\frac{e+x}{4}.$$

The equation of motion along BP is

$$4\frac{d^2x}{dt^2}=4g-T$$

$$=4g-\lambda\,.\,\frac{e+x}{4}$$

$$=4g-12g\,.\,\frac{\frac{4}{3}+x}{4}$$

$$=4g-(4g+3gx)$$

$$=-3gx.$$

$$\therefore \qquad \frac{d^2x}{dt^2}=-24x$$

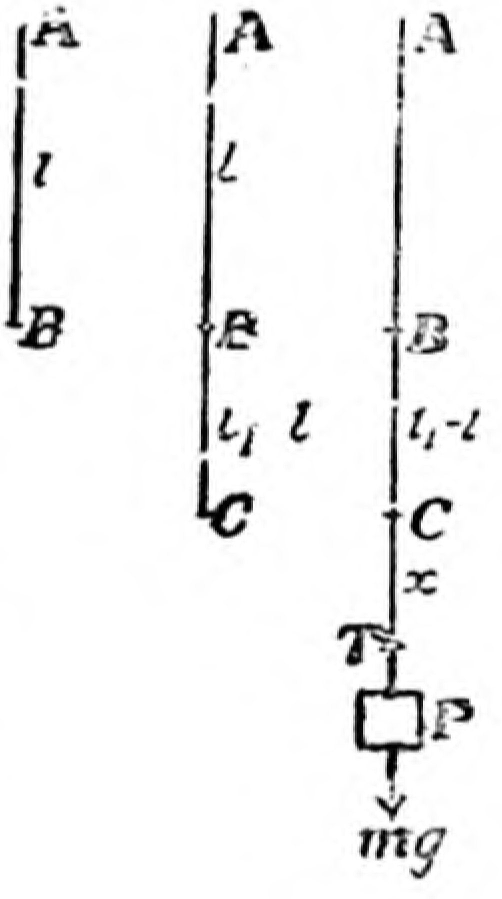
which shows that the motion is simple harmonic about B, the equilibrium position.

Here $\mu=24$.

$\therefore$ Periodic time $=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{24}}=\frac{\pi}{\sqrt{6}}$ sec.

Ex. 14. *A light elastic string of natural length l has one extremity fixed at a point A and the other attached to a stone the weight of which, in equilibrium, would extend the string to a length l_1. Show that if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{l_1^2-l^2}$ below the equilibrium position.* (*P.U.*)

Sol. Let m be the mass of the stone attached to the point B of the string whose upper end is fixed.



Then the extension produced $=l_1-l$.

In the equilibrium position,

$$mg=\lambda\frac{l_1-l}{l}$$

$$\therefore \qquad \lambda=\frac{mgl}{l_1-l}.$$

The particle falls freely from A under gravity, acquiring a velocity $=\sqrt{2gl}$ at B. The string is then taut. Let P be its position at time t at a distance x from C, the equilibrium position.

Then the tension in the string in this position is given by

$$T=\lambda\frac{(l_1-l)+x}{l}$$

$$=\frac{mgl}{l_1-l}\;\frac{(l_1-l)+x}{l}$$

$$=mg+\frac{mgx}{l_1-l}.$$

The equation of motion is

$$mv\frac{dv}{dx}=mg-T$$

$$=mg-mg-\frac{mgx}{l_1-l}$$

$$= -\frac{mg}{l_1 - l} \cdot x$$

or
$$v\frac{dv}{dx} = -\frac{g}{l_1 - l}\, x$$

which shows that the motion is Simple Harmonic about C, the equilibrium position.

Integrating, we get

$$\frac{v^2}{2} = -\frac{g}{l_1 - l} \cdot \frac{x^2}{2} + C,$$

where C is a constant.

At B,
$$x = -(l_1 - l) \text{ and } v = \sqrt{2gl}$$

$$\therefore \quad \frac{2gl}{2} = -\frac{g}{l_1 - l} \cdot \frac{(l_1 - l)^2}{2} + C$$

$$\therefore \quad C = gl + \frac{g(l_1 - l)}{2}$$

$$\therefore \quad \frac{v^2}{2} = -\frac{g}{l_1 - l} \cdot \frac{x^2}{2} + gl + \frac{g(l_1 - l)}{2}$$

when $v=0$, the value of x from this equation gives the amplitude.

$$\therefore \quad 0 = -\frac{g}{l_{.} - l} \cdot \frac{x^2}{2} + gl + \frac{g(l_1 - l)}{2}$$

or
$$\frac{g}{l_1 - l} \cdot \frac{x^2}{2} = gl + \frac{g(l_1 - l)}{2}$$

or
$$\frac{gx^2}{l_1 - l} = g(l_1 + l)$$

or
$$x^2 = l_1^2 - l^2$$

$$\therefore \quad x = \sqrt{l_1^2 - l^2}.$$

Ex. 15. *A light elastic string AB of length l is fixed at A, and if a mass m be attached to B, the string would be stretched to double its length. If the mass $\frac{m}{4}$ be attached to B and let fall from A, prove that*

(*i*) *the motion is S.H.M.*

(*ii*) *the amplitude is $\frac{3}{4}l$*

(*iii*) *the period of oscillation is*

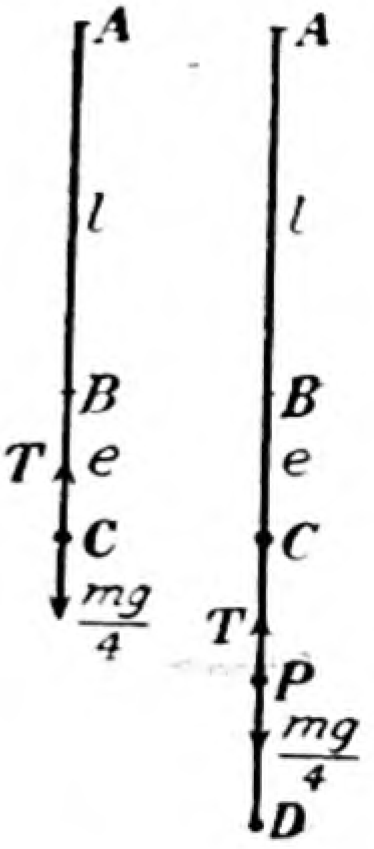
$$\sqrt{\frac{l}{g}}\left[2\sqrt{2} + \frac{\pi}{2} + \sin^{-1}\tfrac{1}{3}\right].$$

(*P.U. T.D.C. II 1966*)

Sol. Since the string is stretched to double its natural length when a mass m is attached, we have

$$mg=\lambda\,\frac{2l-l}{l}$$

$$=\lambda.$$



Let now mass $\frac{m}{4}$ be attached to the end B of the string whose natural is AB$=l$. Let C be the position of equilibrium, where BC$=e$. Then

$$\frac{mg}{4}=\lambda\,.\,\frac{e}{l}$$

$$=mg\,.\,\frac{e}{l}$$

$$\therefore \qquad e=\frac{l}{4}.$$

Since the particle falls freely under gravity from A, the velocity of the particle at B$=\sqrt{2gl}$.

When the particle falls below B, the string becomes tight. Let D be the position of rest. Then obviously D is the lowest point. Let P be the position of the particle at a distance x below C, the equilibrium position.

Then
$$\text{T}=\lambda\,\frac{e+x}{l}$$

$$=mg\,\frac{\frac{l}{4}+x}{l}$$

$$=\frac{mg}{4}+\frac{mgx}{l}.$$

The equation of motion is

$$\frac{m}{4}\cdot\frac{d^2x}{dt^2}=\frac{m}{4}g-\text{T}$$

$$=\frac{m}{4}g-\left(\frac{mg}{4}+\frac{mgx}{l}\right)$$

$$=-\frac{mg}{l}\,.\,x$$

$$\therefore \qquad \frac{d^2x}{dt^2}=-\frac{4g}{l}\,.\,x,$$

which shows that the motion is simple harmonic about C, the position of equilibrium.

Here
$$\mu=\frac{4g}{l}.$$

Let a be the amplitudes and v, the velocity at any instant.

Then
$$v^2=\mu(a^2-x^2)$$
$$=\frac{4g}{l}(a^2-x^2).$$

At
$$B,\ x=\frac{l}{4} \text{ and } v=\sqrt{2gl}.$$

$$\therefore \quad 2gl=\frac{4g}{l}\left(a^2-\frac{l^2}{16}\right)$$

$$\therefore \quad \frac{l^2}{2}=a^2-\frac{l^2}{16}$$

or
$$a^2=\frac{9l^2}{16}$$

$$\therefore \quad a=\tfrac{3}{4}l$$

$$\therefore \quad \text{amplitude}=CD=\tfrac{3}{4}l.$$

Now the time $(=t_1)$ taken by the particle from D to C
$$=\tfrac{1}{4}\times\text{period of S.H.M.}$$
$$=\frac{1}{4}\cdot\frac{2\pi}{\sqrt{\mu}}$$
$$=\frac{1}{4}\cdot\frac{2\pi}{\sqrt{4g/l}}$$
$$=\frac{\pi}{4}\sqrt{\frac{l}{g}}.$$

Time $(=t_2)$ taken by the particle from C to B
$$x=a\sin\sqrt{\mu}\,t_2$$

where
$$x=\frac{l}{4}.$$

$$\therefore \quad \frac{l}{4}=\frac{3l}{4}\sin\sqrt{\frac{4g}{l}}\,.\,t_2$$

$$\therefore \quad t_2=\sqrt{\frac{l}{4g}}\sin^{-1}\frac{1}{3}.$$

Now since amplitude $CD(=\tfrac{3}{4}l)$ is greater than the length $BC(=l/4)$, the motion above B is freely due to gravity.

$\therefore$ time $(=t_3)$ taken by the particle from B to A
$$=\frac{\text{Velocity at B}}{g}$$
$$=\frac{\sqrt{2gl}}{g}=\sqrt{\frac{2l}{g}}.$$

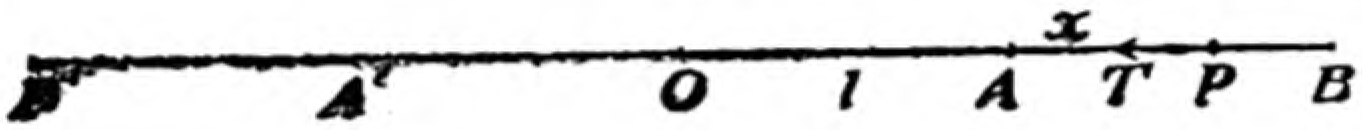
$\therefore$ time for one oscillation

$$=2[t_1+t_2+t_3]$$

$$=2\left[\frac{\pi}{4}\sqrt{\frac{l}{g}}+\frac{l}{2\sqrt{g}}\sin^{-1}\left(\frac{1}{3}\right)+\sqrt{2}\sqrt{\frac{l}{g}}\right]$$

$$=\sqrt{\frac{l}{g}}\left[\frac{\pi}{2}+2\sqrt{2}+\sin^{-1}\frac{1}{3}\right].$$

4·7 Horizontal Elastic String : Let one end of an elastic string of natural length l be fixed at a point O on a smooth horizontal table. Let a particle of mass m be attached to the other end of the string. If now the particle is displaced along the line OA to B where AB$=a$ and then let go and if P be the position of the particle at time t so that AP$=x$, then the equation of motion along AB is



$$m\frac{d^2x}{dt^2}=-\text{T}$$

$$=-\lambda\frac{x}{l}$$

or $$\frac{d^2x}{dt^2}=-\frac{\lambda}{ml}\cdot x \qquad \ldots(1)$$

which shows that the motion is simple harmonic about A.

$$\text{Period of oscillation}=\frac{2\pi}{\sqrt{\frac{\lambda}{ml}}}$$

$$=2\pi\sqrt{\frac{lm}{\lambda}} \qquad \ldots(2)$$

$\therefore$ time from B to A$=\frac{1}{4}\cdot 2\pi\frac{lm}{\lambda}=\frac{\pi}{2}\sqrt{\frac{lm}{\lambda}}.$

When the particle reaches A, the string is no longer stretched and the simple harmonic motion ceases. Due to the velocity acquired at A, the particle moves further till it comes to A′ on the other side of O such that OA′$=$OA$=a$. The velocity from A to A′ is constant since the string becomes slack. Now as the particle goes beyond A′, the string becomes taut so that motion is again simple harmonic till the particle reaches the point B′, where OB′$=$OB. The particle is now instantaneously at rest. It then moves towards O under S.H.M. upto A′ and then with uniform velocity upto A and then again with S.H.M. upto B. At this point the particle is again at rest. This motion is repeated.

Now equation (1) can be written as

$$v\frac{dv}{dx}=-\frac{\lambda}{lm}x.$$

Integration gives

$$\frac{v^2}{2}=-\frac{\lambda}{lm}\ \frac{x^2}{2}+\mathrm{C}$$

At B, $v=0$ and $x=a$

$$\therefore \qquad \mathrm{C}=\frac{\lambda}{2lm}a^2$$

$$\therefore \qquad v^2=\frac{\lambda}{lm}(a^2-x^2)$$

$$\therefore \qquad v=-\sqrt{\frac{\lambda}{lm}}\sqrt{a^2-x^3}$$

negative sign being taken because x decreases as t increases so that $\frac{dx}{dt}$, *i.e.*, v is —*ve*.

Velocity at A($x=0$) is $\sqrt{\frac{\lambda}{lm}}\,a$ in the direction AO.

The particle moves with uniform velocity from A to A′, where AA′$=2l$.

$$\therefore \text{ time taken from A to A}'=\frac{2l}{a\sqrt{\frac{\lambda}{lm}}}=\frac{2l}{a}\sqrt{\frac{lm}{\lambda}}.$$

Motion from A′ to B′ is again due to simple harmonic motion and time taken

$$=\frac{\pi}{2}\sqrt{\frac{lm}{\lambda}}.$$

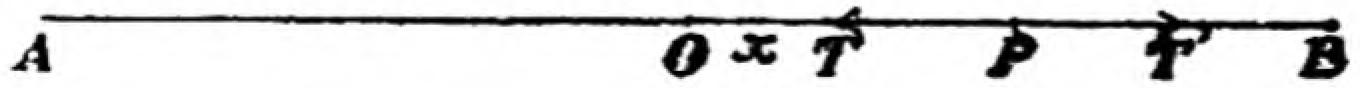
$$\therefore \text{ time from B to B}'=\frac{\pi}{2}\sqrt{\frac{lm}{\lambda}}+\frac{2l}{a}\sqrt{\frac{lm}{\lambda}}+\frac{\pi}{2}\sqrt{\frac{lm}{\lambda}}$$

$$=\sqrt{\frac{lm}{\lambda}}\left(\pi+\frac{2l}{a}\right).$$

$\therefore$ the particle returns to the point A after time

$$=2\sqrt{\frac{lm}{\lambda}}\left(\pi+\frac{2l}{a}\right).$$

Ex. 16. *An elastic string to the middle point of which a particle of mass m is attached is stretched to twice its natural length and placed on a smooth horizontal table and its ends are then fixed. Find the period of oscillation when the particle is displaced in the direction of the string.* (*P.U. B.Sc. 1960*)



Sol. Let $2l$ be the natural length of the string. Then the extended length is $2l$. Let O be the middle point and A and B the fixed ends. Then AO$=$OB$=2l$.

Let P be any position of the particle, where $OP=x$.

Then $$AP=AO+OP$$
$$=2l+x.$$

$\therefore$ extension of $AP=2l+x-l$
$$=l+x,$$

and $$BP=OB-OP=2l-x$$

$\therefore$ extension of $BP=2l-x-l$
$$=l-x.$$

Let T and T′ be the tensions in the portions PA and PB respectively.

Then $$T=\lambda\frac{l+x}{l}$$

and $$T'=\lambda\frac{l-x}{l}.$$

The equation of motion of P is
$$m\frac{d^2x}{dt^2}=T'-T$$
$$=\lambda\frac{l+x}{l}-\lambda\frac{l-x}{l}$$
$$=-\lambda\frac{2x}{l}$$

or $$\frac{d^2x}{dt^2}=-\frac{2\lambda}{lm}x$$

which shows that the motion is simple harmonic about O.

$\therefore$ $$\text{period}=\frac{2\pi}{\sqrt{\frac{2\lambda}{lm}}}=\pi\sqrt{\frac{2lm}{\lambda}}.$$

Exercise IV (C)

1. A body of mass 5 lb. is hung by a light extensible string and is found to stretch it 6 inches. The mass is then pulled down a further distance of 2 inches and released. Find the period of oscillations. (*P.U.*)

2. A spiral string of length one foot is stretched to 13 inches when supporting a four ounce weight. Determine the period of vertical obscillation of a three-ounce weight supported by the same spring. (*P.U.*)

3. If t_1 and t_2 be the periods corresponding to two different attached weights and c_1 and c_2 the statical extensions due to these weights, prove that

$$g=\frac{4\pi^2(c_1-c_2)}{t_1^2-t_2^2}.$$

4. An elastic string without weight, of which the unstretched length is l and the modulus of elasticity is the weight of n ozs. is suspended by one end and the mass of m ozs. is attached to the other ; show that the time of small vertical oscillation is

$$2\pi\sqrt{\frac{ml}{ng}}.$$

5. A particle of mass m is attached to one end of an elastic string, of natural length a and modulus of elasticity $2mg$, whose other end is fixed at O. The particle is let fall from A, where A is vertically above O and OA$=a$. Show that its velocity will be zero at B, where OB$=3a$. Calculate also time from A to B.

6. An elastic string of natural length $2l$ can just support a certain weight when it is stretched till its whole length is $3l$. One end of the string is now attached to a point on a smooth horizontal table and the same weight is attached to the other end. Prove that if the weight is pulled to any distance and then let go, the string will become slack after a time

$$\tfrac{1}{2}\pi\sqrt{\frac{l}{g}}. \qquad (P.U.)$$

7. An elastic string supporting a heavy particle hangs in equilibrium, the elongation produced by the weight of the particle being equal to e. The particle is now pulled down below the equilibrium position through a distance d greater than e, and then released. Show that the height to which the particle will rise after the string ceases to be taut is $\frac{d^2-e^2}{2e}$.

8. State Hooke's Law. Two bodies of masses M and M′ are attached to the lower end of an elastic string whose upper end is fixed and hang at rest. If M′ falls off, and a be the unstretched length of the string, and b and c the distances by which it would be stretched when supporting M and M′ respectively, find at time t the distance of M from the upper end of the string. (*P.U.*)

9. One end of an elastic string (modulus of elasticity λ) whose natural length is a is fixed to a point on a horizontal table and the other end is tied to a particle of mass m, which is lying on the table. The particle is pulled to a distance from the point of attachment

of the string equal to twice its natural length and then let go. Show that the time of complete oscillation is

$$2(\pi+2)\sqrt{\frac{ma}{\lambda}}.$$

Answers

1. $T=\frac{\pi}{4}$ sec. **2.** $\frac{\pi}{8\sqrt{2}}$ sec.

5. $\frac{\pi}{16}+\frac{1}{8}\sin^{-1}\frac{1}{3}+\frac{a}{2\sqrt{2}}$ seconds.

8. $a+b+c\cos\sqrt{\frac{g}{b}}\,t.$

Miscellaneous Exercises on Chapter IV

1. A particle starts from rest and moves along a st. line with an acceleration f varying as t^n. If v be its velocity at a distance s from the starting point, show that

$$(n+1)v^2=(n+2)fs.$$

2. A particle moves in a straight line under a force to a point in it varying as (distance)$^{-\frac{4}{3}}$: show that the velocity in falling from rest at infinity to a distance a is equal to that acquired in falling from rest a distance a to a distance $\frac{1}{8}a$. *(Lucknow)*

3. A steam-boat is moving with velocity V when steam is shut off. Given that the retardation at any subsequent time is equal to the magnitude of the velocity at that time, find the velocity and distance travelled in time t after the steam is shut off.

4. A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes in reaching a height h is

$$\frac{1}{3}\sqrt{\frac{2a}{g}}\left[\left(1+\frac{h}{a}\right)^{\frac{3}{2}}-1\right]$$

where a is the radius of the earth.

5 The acceleration of a particle moving in a straight line is always directed towards a fixed point O in the line and varies as the square of its distance from O. It starts from rest from a point at a distance c from O. Show that the time the particle takes to reach the centre O is $\frac{\pi}{\sqrt{\mu}}\left(\frac{c}{2}\right)^{3/2}$; μ being the constant of variation.

6. A particle moves in a straight line from a distance a, to centre of force. Show that the time required to reach the centre is

$$\frac{a^2}{\sqrt{\mu}} \text{ or } \frac{2a^{4/3}}{\sqrt{3\mu}}$$

according as the force of acceleration is μr^{-3} or $\mu r^{-\frac{5}{3}}$.

7. A particle of mass m is attracted towards a fixed point O with a force $m\mu$ times the distance from O. If initially it is projected towards O with a velocity v from a point distant c from O, show that the amplitude of its oscillation is

$$\sqrt{c^2+\frac{v^2}{\mu}}\cdot \qquad \textit{(P.U.)}$$

8. If a is the amplitude and T the period of a S.H.M., show that $v^2T^2=4\pi^2(a^2-x^2)$, where v is the velocity at any distance x. When the particle is at a distance $\frac{3a}{5}$ from the mean position, its velocity is increased to five times. Find the new amplitude. What would be the new amplitude if the velocity were decreased to one-fifth ?

9. A particle of mass 2 gm. is moving with S.H.M. If its greatest velocity is 20π cm. per sec. and the amplitude of oscillation is 20 cm., find the period and the force of attraction towards the centre when the particle is at its greatest distance from it.

(P.U. 1961)

10. A mass of 1 gm. vibrates through 1 mm. on each side of the middle point of its path 256 times per second ; assuming the motion to be S.H.M., show that the maximum force upon the particle is $\frac{1}{10}(512\pi)^2$ dynes. *(P.U.)*

11. A particle rests in equilibrium under the attraction of two centres of forces which attract directly as the distance, their attraction per unit of mass at unit distance being μ and μ'. The particle is slightly displaced towards one of them, show that the time of small oscillation is

$$\frac{2\pi}{\sqrt{\mu+\mu'}}.$$

12. An elastic string of natural length $a+b$ $(a>b)$ and modulus of elasticity λ has a particle of mass m attached to it at a distance a from one end, which is fixed to a point A of a smooth horizontal plane. The other end of the string is fixed to a point B so that the string is just unstretched. If the particle be held at B and then released, show that it will oscillate to-and-fro through a distance $\frac{b(\sqrt{a}+\sqrt{b})}{\sqrt{a}}$ in a periodic time $\pi(\sqrt{a}+\sqrt{b})\sqrt{\frac{m}{\lambda}}$.

13. A particle of mass m executes S.H.M. in the line joining the points A and B on a smooth table and is connected with these points by elastic string whose tensions in equilibrium are each T ; show that the time of an oscillation is

$$2\pi\sqrt{\frac{mll'}{T(l-l')}}$$

where l, l' are the extensions of the strings beyond their natural lengths.

14. The particles of masses m_1 and m_2 are tied to the ends of an elastic string of natural length a and modulus λ. They are placed on a smooth plane so that the string is just taut and m_2 is projected with any velocity directly away from m_1. Prove that the string will become slack after the lapse of time

$$\pi\sqrt{\frac{am_1m_2}{\lambda(m_1+m_2)}}.$$ *(P.U. 1969 S)*

Answers Misc. Exercise IV

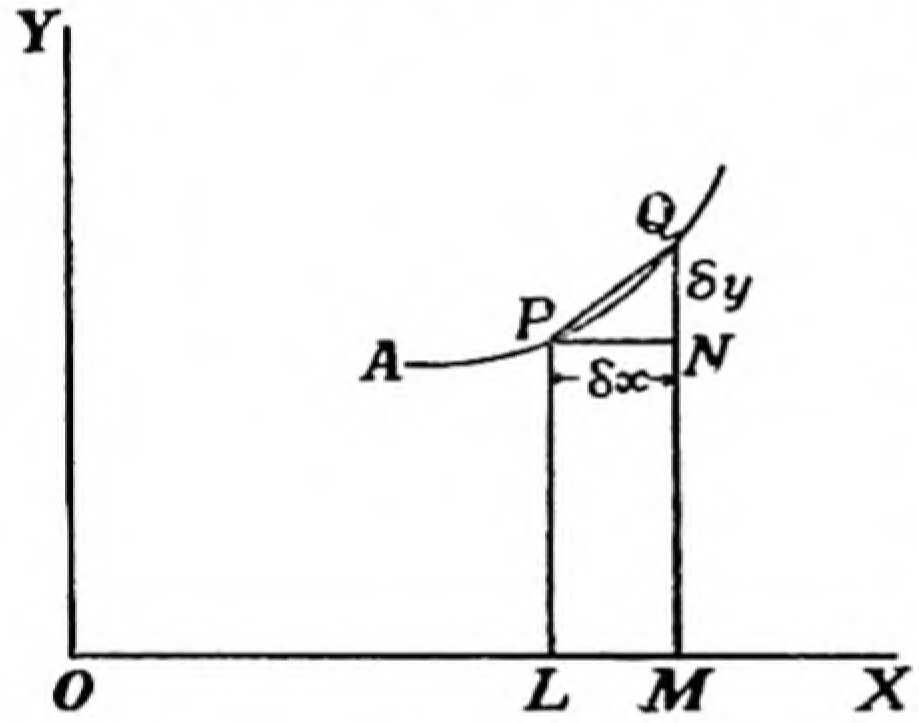
3. $\frac{V}{e}$; $V\left(\frac{e-1}{e}\right)$.   **10.** T=2 sec. ; F=40 π^2 dynes.

CHAPTER V

PROJECTILES

5·1 Motion in a plane. We now discuss the motion of a particle in a plane. In this case the position of the particle in the plane is known if we know its co-ordinates referred to two fixed perpendicular axes OX and OY in the plane. It will be thus convenient to find the expressions for the components of the velocity and acceleration along these perpendicular directions.

5·11 Components of velocity along OX and OY.



Let the particle be moving along the curve APQ and let P, Q be its positions at times t and $t+\delta t$. If (x, y) and $(x+\delta x, y+\delta y)$ be the co-ordinates of P and Q referred to the rectangular axes OX and OY, then the components of the displacement PQ along the perpendicular directions OX and OY are δx, δy respectively.

∴ the resolved part of the velocity parallel to OX

$$=\underset{\delta\to 0}{\mathrm{Lt}}\ \frac{\delta x}{dt}=\frac{dx}{dt} \qquad \ldots(\mathrm{I})$$

and the resolved part of the velocity parallel to OY

$$=\underset{\delta t\to 0}{\mathrm{Lt}}\ \frac{\delta y}{\delta t}=\frac{dy}{dt}. \qquad \ldots(\mathrm{II})$$

If the resultant velocity be denoted by V,

then
$$\mathrm{V}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\frac{ds}{dt}. \qquad \ldots(\mathrm{III})$$

(*From Differential Calculus*)

Let θ be the angle which the direction of motion makes with the axis of x.

Then
$$\tan\theta=\frac{dy}{dt}\Big/\frac{dx}{dt}$$
$$=\frac{dy}{dx}$$
$$=\text{slope of the tangent at P} \qquad \ldots(\mathrm{IV})$$

∴ the direction of motion of the particle is along the tangent to the path.

5·12 Another method for the resultant velocity at P.

Let A be a fixed point on the curve and let arcs AP and AQ be s and $s+\delta s$ respectively. Let chord PQ$=\delta c$.

Then
$$V=\underset{\delta t\to 0}{\mathrm{Lt}}\ \frac{\text{chord PQ}}{\delta t}$$
$$=\underset{\delta t\to 0}{\mathrm{Lt}}\ \frac{\text{chord PQ}}{\text{arc PQ}}\cdot\frac{\text{arc PQ}}{\delta t}=\underset{Q\to P}{\mathrm{Lt}}\ \frac{\delta c}{\delta s}\ \underset{\delta t\to 0}{\mathrm{Lt}}\ \frac{\delta s}{\delta t}$$
$$=1\times\frac{ds}{dt}=\frac{ds}{dt}\cdot$$

5·13 Components of acceleration along OX, OY.

Let u, v denote the resolved parts of the velocity of the particle parallel to the axes at time t and $u+\delta u$, $v+\delta v$ be the resolved parts at time $t+\delta t$.

Then the average rates of change of the component velocities are
$$\frac{\delta u}{\delta t} \text{ and } \frac{\delta v}{\delta t}\cdot$$

$\therefore$ resolved part of acceleration at P along OX
$$=\underset{\delta t\to 0}{\mathrm{Lt}}\ \frac{\delta u}{\delta t}=\frac{du}{dt}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{d^2x}{dt^2} \qquad \ldots\text{(I)}$$

Also
$$\frac{du}{dt}=\frac{du}{dx}\cdot\frac{dx}{dt}=u\,\frac{du}{dx} \qquad \ldots\text{(II)}$$

And resolved part of acceleration at P along OY
$$=\underset{\delta t\to 0}{\mathrm{Lt}}\ \frac{\delta v}{\delta t}=\frac{dv}{dt}=\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{d^2y}{dt^2} \qquad \ldots\text{(III)}$$

Also
$$\frac{dv}{dt}=\frac{dv}{dy}\cdot\frac{dy}{dt}=v\,\frac{dv}{dy}. \qquad \ldots\text{(IV)}$$

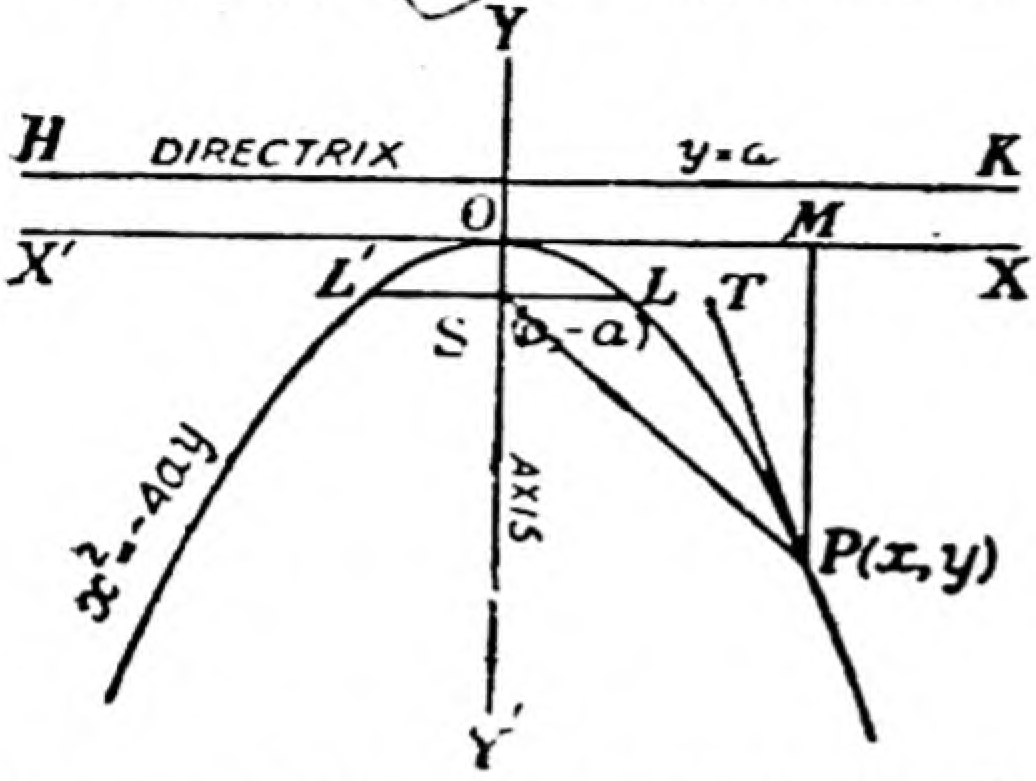
Hence the **resolved parts of the acceleration are**
$$\frac{du}{dt} \text{ or } \frac{d^2x}{dt^2} \text{ or } u\,\frac{du}{dx} \text{ along OX}$$
and
$$\frac{dv}{dt} \text{ or } \frac{d^2y}{dt^2} \text{ or } v\,\frac{dv}{dy} \text{ along OY.}$$

5·14 Equations of motion in a plane. In this case two equations of motion are obtained by resolving the forces along the co-ordinate axes. If P and Q are the sum of the resolved parts of the forces parallel to OX and OY respectively, then the equations of motion, from Newton's second law of motion, parallel to these axes are
$$m\frac{d^2x}{dt^2}=P \qquad \ldots\text{(I)}$$

and $$m\frac{d^2y}{dt^2}=Q \qquad ...(II)$$

where m is the mass of the particle.

5·2 Parabola. The reader has studied in Plane Co-ordinate Geometry, that the equation of the parabola with its vertex uppermost, at the origin, its axis along the y-axis and tangent at the vertex as x-axis is



$$x^2=-4ay.$$

The focus is S(0, $-a$)
Latus Rectum LL$'=4a$.

Equation of the directrix, HK is

$$y=a.$$

Some of the properties of the parabola are :

(1) Distance between the focus and the vertex is $\frac{1}{4}$ (Latus Rectum)$=a$.

(2) Any point on the parabola is equidistant from the focus and the directrix, *i.e.* PS=PM.

(3) Tangents at the ends of a focal chord are perpendicular and intersect on the directrix.

(4) The foot of perpendicular from the focus on any tangent lies on the tangent at the vertex.

(5) The tangent PT at a point P, bisects (externally) the angle between the focal distance of P and the line through P $\parallel$ the axis.

5·21 The Equation $(\mathbf{x}-\mathbf{h})^2=-4\mathbf{a}(\mathbf{y}-\mathbf{k})$. ...(I)

If we shift the origin to the point (h, k) and denote the current co-ordinates by (X, Y) ; then

$$X=x-h \quad \text{and} \quad Y=y-k.$$

So that the equation (I), becomes

$$X^2=-4aY \qquad ...(II)$$

which represents a parabola, whose

(*i*) Axis is X$=0$

or $x-h=0$ ...(III)

(*ii*) Tangent at vertex is

Y$=0$

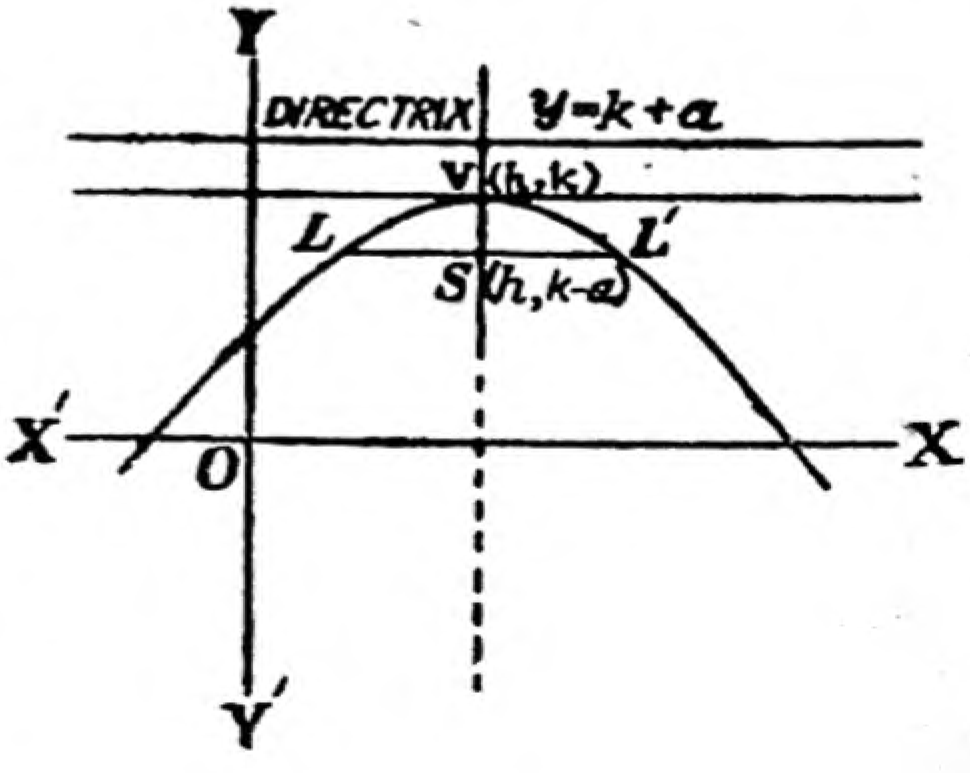
or $y-k=0$ ...(IV)

(*iii*) Vertex, the point of intersection of (III) and (IV) is

V(h, k).

(*iv*) Latus Rectum, LL$'=4a$.

(*v*) Minus sign on R.H.S. shows that concavity of the parabola is along OY$'$, *i.e.* downwards.



(*vi*) Distance of focus below the vertex$=\frac{1}{4}$(L.R.)$=a$.

(*vii*) Focus$\equiv S(h,\ k-a)$.

(*viii*) Directrix has the equation

$$y=k+a$$

Ex. 1. *Find the axis, vertex, focus and directrix of the parabola*

$$4x^2-8x+7y=0. \qquad \text{...(I)}$$

(I) can be rewritten as

$$x^2-2x+1=-\tfrac{7}{4}y+1$$

or
$$(x-1)^2=-\tfrac{7}{4}(y-\tfrac{4}{7}) \qquad \text{...(II)}$$

which is of the form
$$X^2=-4aY \qquad \text{...(III)}$$

where
$$X=x-1,\ Y=y-\tfrac{4}{7}.$$

(II) represents a parabola with concavity downwards, whose

(*i*) Axis is $X=0$ or $x-1=0$.

(*ii*) Tangent at Vertex is $Y=0$ or $y-\frac{4}{7}=0$.

(*iii*) Vertex$\equiv V(1,\ \frac{4}{7})$.

(*iv*) Latus Rectum$=\frac{7}{4}$.

(*v*) Distance of focus below the vertex$=\frac{1}{4}$(L.R.)$=\frac{7}{16}$.

(*vi*) $\therefore$ Focus$\equiv S(1,\ \frac{4}{7}-\frac{7}{16})\equiv(1,\ \frac{15}{112})$.

(*vii*) Equation of Directrix is

$$y=\tfrac{4}{7}+\tfrac{7}{16} \quad \text{or} \quad 113y=112$$

Clearly parabola (I) passes through the origin.

Ex. 2. *Find the axis, vertex, focus and directrix etc. of the parabola whose equation is*

$$y=x\ \tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}. \qquad \text{...(I)}$$

(I) may be written as

$$gx^2-2u^2\cos^2\alpha.\tan\alpha.x=-2u^2\cos^2\alpha.y$$

or
$$x^2-\frac{2u^2\sin\alpha.\cos\alpha}{g}\ x+\frac{u^4\sin^2\alpha\cos^2\alpha}{g^2}$$
$$=-\frac{2u^2\cos^2\alpha}{g}\ y+\frac{u^4\sin^2\alpha\cos^2\alpha}{g^2}$$

or
$$\left(x-\frac{u^2\sin\alpha\cos\alpha}{g}\right)^2=-\frac{2u^2\cos^2\alpha}{g}\left(y-\frac{u^2\sin^2\alpha}{2g}\right) \qquad \text{...(II)}$$

which is of the form
$$X^2=-4aY \qquad \text{...(III)}$$

where
$$X=x-\frac{u^2\sin\alpha\cos\alpha}{g}\ ;\ Y=y-\frac{u^2\sin^2\alpha}{2g}.$$

∴ (II) represents a parabola with concavity downwards, whose

(*i*) Axis is $X=0$ or $x=\dfrac{u^2 \sin\alpha \cos\alpha}{g}$.

(*ii*) Tangent at vertex $Y=0$ or $y=\dfrac{u^2 \sin^2\alpha}{2g}$.

(*iii*) Vertex $\equiv\left(\dfrac{u^2 \sin\alpha \cos\alpha}{g},\ \dfrac{u^2 \sin^2\alpha}{2g}\right)$.

(*iv*) L.R. $=\dfrac{2u^2 \cos^2\alpha}{2g}$.

(*v*) Distance of the focus below the vertex

$$=\frac{1}{4}(\text{L.R.})=\frac{u^2 \cos^2\alpha}{2g}.$$

(*vi*) Ordinate of the focus$=\dfrac{u^2 \sin^2\alpha}{2g}-\dfrac{u^2 \cos^2\alpha}{2g}=\dfrac{-u^2 \cos 2\alpha}{2g}$.

$$\therefore \quad \text{Focus}\equiv\left(\frac{u^2 \sin\alpha \cos\alpha}{g},\ -\frac{u^2 \cos 2\alpha}{2g}\right)$$

$$\equiv\left(\frac{u^2 \sin 2\alpha}{2g},\ -\frac{u^2 \cos 2\alpha}{2g}\right).$$

(*vii*) Directrix is parallel to the tangent at vertex and above it at a height of $\frac{1}{4}$(L.R.).

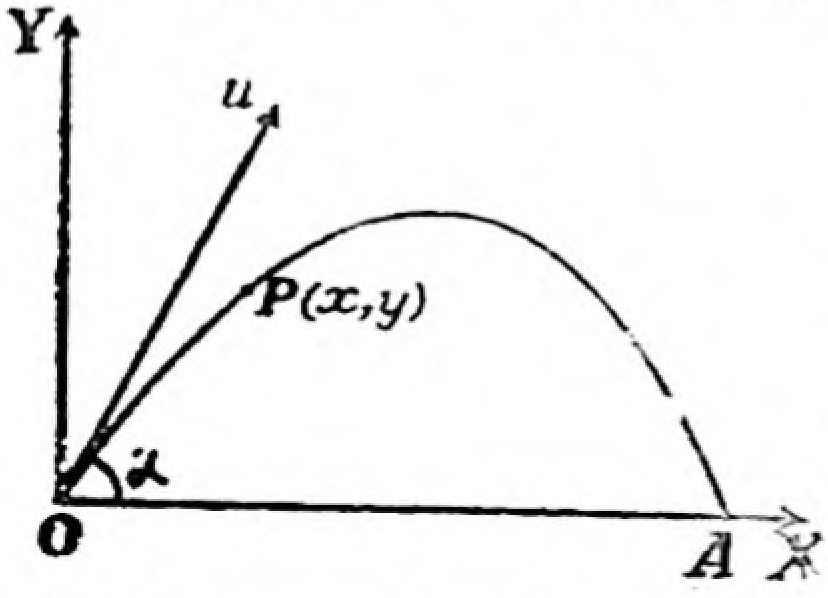
∴ Equation of directrix is $y=\dfrac{u^2 \sin^2\alpha}{2g}+\dfrac{u^2 \cos^2\alpha}{2g}=\dfrac{u^2}{2g}$.

The given parabola (I) clearly passes through the origin.

5·3 Projectile. We shall consider the motion of a particle projected into the air in a direction oblique to the vertical. Such a particle is called a **projectile**. We shall assume that there is no resistance due to the air and that the value of g remains constant. The path described by the particle is called the **Trajectory**.

5·31 *A particle of mass m is projected in vacuum with velocity u, in a direction making an angle α with the horizontal, to find its motion and the path described.*

Take O, the point of projection, as the origin and the horizontal and the vertical lines through O as the axes of x and y.



The resolved parts of the initial velocity u are $u\cos\alpha$ and $u\sin\alpha$ along OX and OY respectively.

Let P (x, y) be the position of the particle after time t.

The only force acting on the particle is its weight mg acting vertically downwards, the resistance of air being neglected.

$\therefore$ the equations of motion

parallel to x-axis

$$m\frac{d^2x}{dt^2}=0$$

or

$$\frac{d^2x}{dt^2}=0$$

Integrating, we get :

$$\frac{dx}{dt}=c_1$$

When $t=0$, $\dfrac{dx}{dt}=u\cos\alpha.$

$$\therefore \quad c_1=u\cos\alpha$$

whence

$$\mathbf{\frac{dx}{dt}=u\cos\alpha} \quad \text{(I)}.$$

Integrating again,

$$x=u\cos\alpha\,.\,t+c_2.$$

But when $t=0$, particle is at the origin, *i.e.*, $x=0$.

$$\therefore \quad c_2=0.$$

$$\therefore \quad \mathbf{x=u\cos\alpha.t}. \quad \text{...(II)}$$

parallel to y-axis

$$m\frac{d^2y}{dt^2}=-mg$$

$$\frac{d^2y}{dt^2}=-g.$$

Integrating, we get :

$$\frac{dy}{dt}=-gt+c_3.$$

When $t=0$, $\dfrac{dy}{dt}=u\sin\alpha.$

$$\therefore \quad c_3=u\sin\alpha$$

whence

$$\mathbf{\frac{dy}{dt}=u\sin\alpha-gt}. \quad \text{...(III)}.$$

Integrating again,

$$y=u\sin\alpha\,.\,t-\tfrac{1}{2}gt^2+c_4.$$

But when $t=0$, $y=0$.

Hence $c_4=0$.

$$\therefore \quad \mathbf{y=u\sin\alpha.t-\tfrac{1}{2}gt^2}. \quad \text{...(IV)}$$

From equations (I)—(IV) it is clear that the velocity and displacements of the particle along x and y axes can at once be written by assuming that the particle is projected horizontally with a constant velocity $u\cos\alpha$ and vertically upwards with velocity $u\sin\alpha$ under a retardation g.

Eliminating t from (II) and (IV), we get

$$\mathbf{y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}}. \quad \text{...(V)}$$

This is the **equation of the path of the projectile** (or **the trajectory**).

(V) may be rewritten as :

$$x^2-\frac{2u^2\sin\alpha\cos\alpha}{g}x=-\frac{2u^2\cos^2\alpha}{g}y.$$

Making the left hand side a perfect square by adding $\dfrac{u^4\sin^2\alpha\cos^2\alpha}{g^2}$ to both sides, we get

$$\left(x-\frac{u^2\sin\alpha\cos\alpha}{g}\right)^2=-\frac{2u^2\cos^2\alpha}{g}\left(y-\frac{u^2\sin^2\alpha}{2g}\right) \quad \text{...(VI)}$$

which represents a *parabola* with

$$\left(\frac{\mathbf{u^2 \sin \alpha \cos \alpha}}{\mathbf{g}}, \ \frac{\mathbf{u^2 \sin^2 \alpha}}{\mathbf{2g}}\right) \text{ as Vertex} \quad \text{...(VII)}$$

and

$$\text{latus rectum} = \frac{\mathbf{2u^2 \cos^2 \alpha}}{\mathbf{g}}. \quad \text{...(VIII)}$$

Thus the path of the projectile in *vacuum* is a parabola.

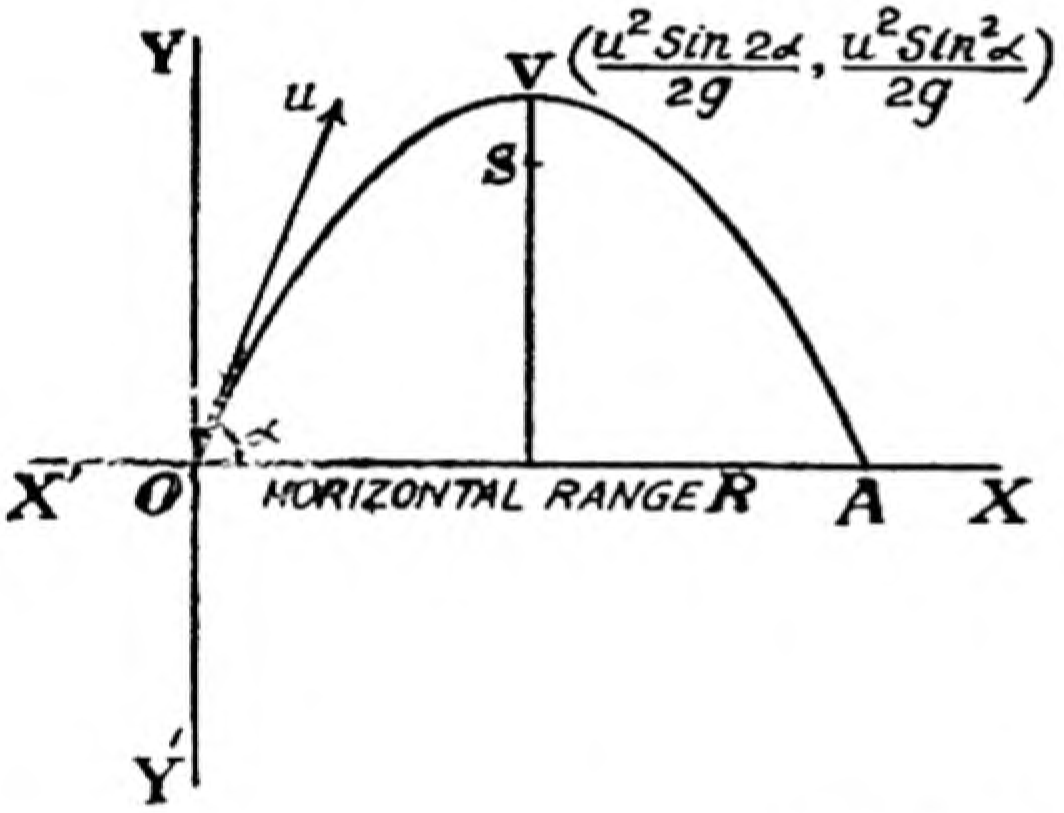
5·32 Important results

(1) **The time of flight T.** It is the time taken by the particle in reaching the horizontal plane through the point of projection.

Since at A, $y=0$

$\therefore$ putting $y=0$ in IV, Art. 5·31,

$$0 = u \sin \alpha t - \tfrac{1}{2} g t^2.$$


Y
u
V
$\left(\frac{u^2 \sin 2\alpha}{2g}, \frac{u^2 \sin^2\alpha}{2g}\right)$
S
α
X'
O
HORIZONTAL RANGE R
A
X
Y'

Rejecting $t=0$, which is the time for O itself

$$\therefore \quad \mathbf{T} = \frac{\mathbf{2u \sin \alpha}}{\mathbf{g}}.$$

(2) **Horizontal Range R.** This is the horizontal distance OA described by the particle in the time of flight.

$\therefore$ R $= u \cos \alpha$. T (From II, Art. 5·31)

$$= \frac{u \cos \alpha . 2u \sin \alpha}{g}$$

or

$$\mathbf{R} = \frac{\mathbf{u^2 \sin 2\alpha}}{\mathbf{g}}$$

which is twice the abscissa of the vertex.

R can also be obtained from (V) (Art. 5·31) by putting $y=0$.

(3) **Maximum Horizontal Range.** R is maximum if

$$\sin 2\alpha = 1 \quad \text{or} \quad \text{if } \alpha = \frac{\pi}{4}.$$

$\therefore$ **the maximum range** $= \frac{\mathbf{u^2}}{\mathbf{g}}$. (*Panjab 1958 S*)

$\therefore$ *if the angle of projection is 45°, the horizontal range is maximum.*

(4) **Directions of projection for a given horizontal range.** (*Panjab 1961*)

Since $\sin 2\alpha = \sin(\pi - 2\alpha) = \sin 2\left(\frac{\pi}{2} - \alpha\right)$.

$$\therefore \quad \text{horizontal range} = \frac{u^2 \sin 2\alpha}{g} = \frac{u^2 \sin 2\left(\frac{\pi}{2} - \alpha\right)}{g}.$$

It follows, therefore, that the angles α and $\frac{\pi}{2} - \alpha$ give the same range.

$\therefore$ *For a given velocity of projection, there are two directions of projection which give the same horizontal range and these two directions are such that the inclination of one with the horizontal is the same as that of the other with the vertical.*

They are also equally inclined to the direction for maximum range.

(5). *Sum of squares of the two times of flight corresponding to two different directions of projection for a given range is* $\frac{4u^2}{g}$. (*P.U. 1961*)

For if α is one direction, the other is $\frac{\pi}{2} - \alpha$.

If t_1, t_2 be the times of flight ; then

$$t_1 = \frac{2u \sin \alpha}{g} \text{ and } t_2 = \frac{2u \sin\left(\frac{\pi}{2} - \alpha\right)}{g} = \frac{2u \cos \alpha}{g}$$

Whence $$t_1^2 + t_2^2 = \frac{4\,u^2}{g^2}.$$

(6) **Greatest height.** At the highest point the vertical component of velocity vanishes.

Using III of Art. 5·31 ; when

$$\frac{dy}{dt} = 0, \quad t = \frac{u \sin \alpha}{g}$$

and from IV,

$$y = u \sin \alpha . \frac{u \sin \alpha}{g} - \frac{1}{2} g \frac{u^2 \sin^2 \alpha}{g^2}$$

$$= \frac{u^2 \sin^2 \alpha}{2g}.$$

The same result could also be obtained by considering that

$$\text{greatest height} = \text{ordinate of the vertex}$$

$$= \frac{u^2 \sin^2 \alpha}{2g}.$$

(7) **Height of the directrix**

$$= \text{height of the vertex} + \tfrac{1}{4} \text{ latus rectum}$$

$$= \frac{u^2 \sin^2 \alpha}{2g} + \frac{1}{4} \cdot \frac{2u^2 \cos^2 \alpha}{g} = \frac{u^2}{2g}$$

which is independent of α, the angle of projection.

Hence all trajectories having the same point of projection and the same initial velocity have a common directrix.

It also follows from the result that the height of the directrix is the same as the height attained by the particle if projected vertically upwards with the same velocity.

(8) **Focus of the Path.**

As found in Example 2 of Art. 5·21,

Focus is $\left(\frac{u^2 \sin 2\alpha}{2g}, -\frac{u^2 \cos 2\alpha}{2g}\right)$.

(9) For the maximum horizontal range,

$$\alpha = 45°.$$

∴ the co-ordinates of the focus are

$$\left(\frac{u^2}{2g}, 0\right)$$

which shows that the focus lies on the horizontal line through the point of projection.

(10) The co-ordinates of the focus are

$$x = \frac{u^2 \sin 2\alpha}{2g}, \quad y = -\frac{u^2 \cos 2\alpha}{2g}.$$

Eliminating α, we get

$$x^2 + y^2 = \frac{u^4}{4g^2}$$

which represents a circle.

Hence the locus of the focus for different directions of projection is a circle having the point of projection as its centre.

(11) Since the horizontal and vertical components of velocity V of projection are V cos α and V sin α and if we denote them by u and v ; $u = V \cos \alpha$; $v = V \sin \alpha$.

∴ Height of Focus above the horizontal through the point of projection

$$= -\frac{V^2 \cos 2\alpha}{2g}$$

$$= \frac{V^2 \sin^2 \alpha - V^2 \cos^2 \alpha}{2g}$$

$$= \frac{v^2 - u^2}{2g}$$ (*P.U. 1958*)

(12) **Vertex of the Path** is given by

$$x = \frac{u^2 \sin \alpha \cos \alpha}{g}, \quad y = \frac{u^2 \sin^2 \alpha}{2g}.$$

If velocity u is given and particles are thrown in different direction, *i.e.*, α is variable, then locus of vertices of different paths got by eliminating α, is $x^2 + 4y^2 = \frac{2u^2}{g} y$ which is an *Ellipse*. (*P.U. 1958*)

Ex. Show that the path of a projectile in vacuum is a parabola if it be projected with a velocity u at an angle α to the horizontal. Find the horizontal range and the time of flight. (*J. & K*)

Ex. A particle is projected with a velocity u at an angle α with the horizontal. Find the greatest height attained by it and show that for a given range, there are two directions of projections. (*J. & K.*)

Ex. 3. *A $\frac{1}{2}$ lb. pebble is projected with a velocity of 100 f.p.s. at an angle of 60° with the horizon, find at the end of 2 seconds :*

(*i*) *the horizontal and vertical distances of the projectile from the point of projection.*

(*ii*) *the resultant velocity of the stone in magnitude and direction.* (*P.U. 1951 S*)

Here $u=100$ f.p.s. ; $\alpha=60°$.

The horizontal and vertical components of velocity after time t are given by

$$\frac{dx}{dt}=u\cos\alpha=50 \qquad \ldots(\mathrm{I})$$

and

$$\frac{dy}{dt}=u\sin\alpha-gt=50\sqrt{3}-32t. \qquad \ldots(\mathrm{II})$$

The displacements in the horizontal and vertical directions are

$$x=u\cos\alpha\,.\,t=50t \qquad \ldots(\mathrm{III})$$

and

$$y=u\sin\alpha\,.\,t-\tfrac{1}{2}gt^2=50\sqrt{3}\,t-16t^2. \qquad \ldots(\mathrm{IV})$$

(*i*) We get x and y for $t=2$, from (III) and (IV) as

$$x=100\text{ ft. and } y=(100\sqrt{3}-64)\text{ ft.}$$

(*ii*) If V be the velocity at $t=2$, making angle θ with the horizontal,

$$\mathrm{V}\cos\theta=50$$
$$\mathrm{V}\sin\theta=50\sqrt{3}-32\times 2=50\sqrt{3}-64.$$

$$\therefore\quad \mathrm{V}=4\sqrt{881-400\sqrt{3}}=4\sqrt{188.2}\text{ f.p.s.}$$

$$\tan\theta=\frac{50\sqrt{3}-64}{50}=\frac{25\sqrt{3}-32}{25}$$

$$\therefore\quad \theta=24°\ 19'\text{ nearly.}$$

Ex. 4. *A projectile is to have a range R on the horizontal and to reach a maximum height h. Find the time of flight, the initial velocity and elevation.*

Also find the maximum range. (*P.U.*)

Let u be the velocity and α the angle of projection.

Then

$$\frac{2u^2\sin\alpha\cos\alpha}{g}=\mathrm{R} \qquad \ldots(\mathrm{I})$$

$$\frac{u^2\sin^2\alpha}{2g}=h \qquad \ldots(\mathrm{II})$$

From (II),

$$u\sin\alpha=\sqrt{2gh} \qquad \ldots(\mathrm{III})$$

Whence from (I),

$$u\cos\alpha=\frac{g\mathrm{R}}{2}\times\frac{1}{\sqrt{2gh}}=\frac{\mathrm{R}\sqrt{g}}{2\sqrt{2h}} \qquad \ldots(\mathrm{IV})$$

Squaring and adding (III) and (IV)

$$u^2=2gh+\frac{gR^2}{8h}=2g\left(h+\frac{R^2}{16h}\right).$$

Dividing (III) by (IV), we get

$$\tan\alpha=\frac{4h}{R} \text{ or } \alpha=\tan^{-1}\left(\frac{4h}{R}\right).$$

Also $$T=\frac{2u\sin\alpha}{g}=2\sqrt{\frac{2h}{2}}.$$

Also Maximum range with the above valocity

$$=\frac{u^2}{g}=2h+\frac{R^2}{8h}.$$

Ex. 5. *Two particles are projected from the same point in the same vertical plane with equal velocities. If t, t′ be the times taken to reach the other common point of their paths and T, T′, the times to the highest point, show that tT+t′T′ is independent of the directions of projection.* *(Raj. 1955 ; Allahabad 1955)*

Let α, α' be the angles of projection. At the common point, if the horizontal distance is x

then $$x=u\cos\alpha t=u\cos\alpha' t'.$$

Also $$T=\frac{u\sin\alpha}{g}$$

and $$T'=\frac{u\sin\alpha'}{g}$$

$$\therefore \quad tT+t'T'=\frac{x}{u\cos\alpha}\cdot\frac{u\sin\alpha}{g}+\frac{x}{u\cos\alpha}\cdot\frac{u\sin\alpha'}{g}$$

$$=\frac{x(\tan\alpha+\tan\alpha')}{g} \qquad \text{...(I)}$$

The equations of the paths are

$$y=x\tan\alpha-\frac{gx^2\sec^2\alpha}{2u^2} \qquad \text{...(II)}$$

and $$y=x\tan\alpha'-\frac{gx^2\sec^2\alpha'}{2u^2} \qquad \text{...(III)}$$

Solving (II) and (III), to get their point of intersection (x, y), we get

$$x(\tan\alpha-\tan\alpha')=\frac{gx^2}{2u^2}(\sec^2\alpha-\sec^2\alpha')$$

$$=\frac{gx^2}{2u^2}(\tan^2\alpha-\tan^2\alpha')$$

or $$\frac{x}{g}(\tan\alpha+\tan\alpha')=\frac{2u^2}{g^2} \qquad \text{...(IV)}$$

$\therefore$ From (I) and (IV)

$$tT+t'T'=\frac{2u^2}{g^2}$$

which is independent of the directions of projection.

Ex. 6. *A gun is fired from a moving platform and the ranges of the shot are observed to be R and S when the platform is moving forwards and backwards respectively with velocity V ; prove that the angle of projection is*

$$\tan^{-1}\left(\frac{g}{4V^2}\cdot\frac{(R-S)^2}{R+S}\right). \qquad (P.U.)$$

Let u be the velocity of projection of the shot relative to the gun and α the angle of projection.

When the platform is moving forwards, the horizontal component of the velocity is $u\cos\alpha+\text{V}$ and when it is moving backwards, the horizontal component is $u\cos\alpha-\text{V}$. The vertical component in both the cases is unaltered and is equal to $u\sin\alpha$.

Now
$$\text{R}=\frac{2\text{ vertical velocity}\times\text{horizontal velocity}}{g}$$

$$=\frac{2\,(u\cos\alpha+\text{V})\,.\,u\sin\alpha}{g} \qquad \ldots(1)$$

and
$$\text{S}=\frac{2\,(u\cos\alpha-\text{V})\,u\sin\alpha}{g} \qquad \ldots(2)$$

From (1) and (2), by adding and subtracting,

$$\text{R}+\text{S}=\frac{4\,u\sin\alpha}{g}\,.\,u\cos\alpha \qquad \ldots(3)$$

$$\text{R}-\text{S}=\frac{4\,u\sin\alpha}{g}\,.\,\text{V} \qquad \ldots(4)$$

Squaring (4) and dividing by (3), we get

$$\frac{(\text{R}-\text{S})^2}{\text{R}+\text{S}}=\frac{16\,u^2\sin\alpha^2\,.\,\text{V}^2}{g^2}\cdot\frac{g}{4\,u^2\sin\alpha\cos\alpha}$$

$$=\frac{4\,\text{V}^2}{g\tan\alpha}$$

$$\therefore \qquad \tan\alpha=\frac{g}{4\text{V}^2}\cdot\frac{(\text{R}-\text{S})^2}{\text{R}+\text{S}}$$

$$\therefore \qquad \alpha=\tan^{-1}\left(\frac{g}{4\text{V}^2}\cdot\frac{(\text{R}-\text{S})^2}{\text{R}+\text{S}}\right).$$

Exercise V A

1. What is the least velocity of projection requird to obtain a horizontal range of 200 ft., and what will be the time of flight ?

2. A body is projected at such an angle that the horizontal range is three times the greatest height. Find the angle of projection, and if, with this angle the range is 400 yards, find the necessary velocity of projection and the time of flight.

3. Prove that if the time of flight of a bullet over a horizontal range R is T seconds the inclination of the direction of projection to the horizontal is $\tan^{-1}\left(\frac{g\text{T}^2}{2\text{R}}\right)$. *(P.U. 1963)*

4. A particle is projected at an angle of elevation $\sin^{-1}\frac{4}{5}$, and its range on the horizontal plane is 4 miles, find the velocity of projection and the velocity at the highest point of its path.

5. A pilot is flying with a speed of 90 miles per hour at a height of 576 feet. Neglecting air resistance find how far from his target must he release a bomb in order that it may hit the target. *(P.U. 1965)*

6. If the focus of a trajectory lies as much below the horizontal plane through the point of projection as the vertex is above, prove that the angle of projection is $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$. *(P.U.)*

7. A ball is thrown from the top of a tower 200 ft. high with a velocity of 80 ft./sec. at an elevation of 30° above the horizon. Find the horizontal distance from the foot of the tower of the point where it hits the ground. *(P.U.)*

8. If t be the time in which a projectile reaches a point P in its path and t' the time from P till it reaches the horizontal plate through the point of projection, show that the height of P above the horizontal plane is $\frac{1}{2}gtt'$.

9. (*a*) A particle has initially a velocity whose horizontal and vertical components are u and v respectively. If R is the range and h the greatest height attained show that

$$\frac{4h}{R}=\frac{v}{u} \quad \text{and} \quad \frac{8h}{g}=\left(\frac{R}{u}\right)^2.$$

(*b*) If r be the horizontal range of a projectile and h its greatest height, prove that the initial speed is $2g\left(h+\frac{g^2}{16h}\right)^{\frac{1}{2}}$. *(J. & K.)*

10. Obtain the equation of the path of a projectile and show that it may be written in the form :

$$\frac{yR}{x(R-x)}=\tan\alpha,$$

where R is the horizontal range, and α the angle of projection.

Deduce that the greatest height attained by the projectile is $\frac{R\tan\alpha}{4}$ and that this occurs half-way. *(P.U.)*

11. If any number of particles are projected simultaneously from the same point with equal velocities in the same vertical plane, prove that at any instant all the particles will be on a circle of radius ut.

Show also that the centre of the circle descends with acceleration g. *(P.U.)*

12. Particles are projected simultaneously in the same vertical plane from the same point. Show that the locus of the foci of all the trajectories is a parabola when

(*i*) the horizontal component of the velocity for each is the same ;

(*ii*) the initial vertical component of the velocity is the same ;

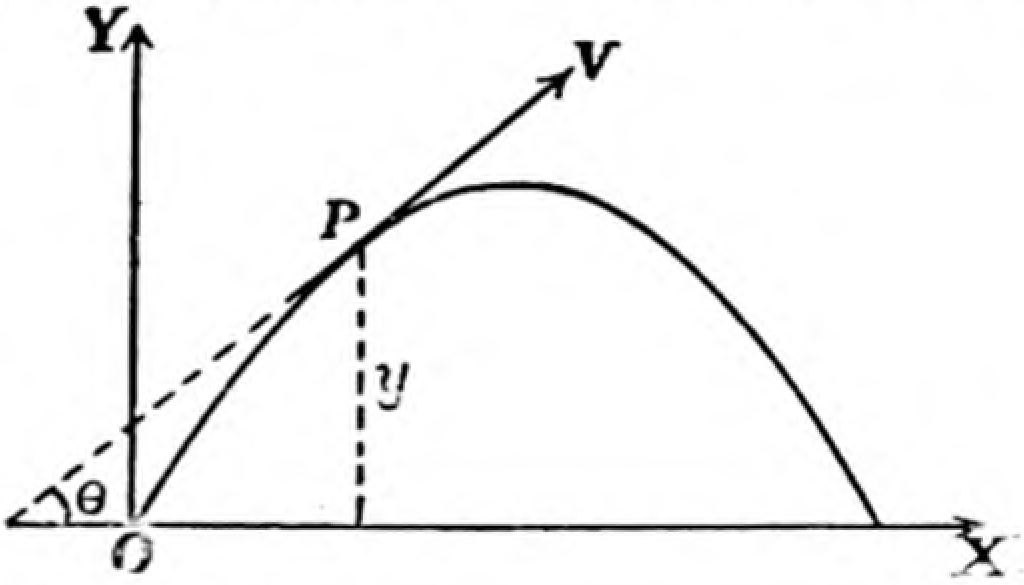
(*iii*) the time of flight for each is the same.

13. The range of a rifle bullet is 1200 yards when α is the elevation of projection. Show that if the rifle is fired from a car travelling at 10 miles/hour towards the target, the range will be increased by $220\sqrt{\tan \alpha}$ feet.

14. A shell bursts on contact with the ground and pieces from it fly in all directions with all velocities upto 80 ft. per second. Show that a man 100 feet away is in danger for $\frac{5}{\sqrt{2}}$ seconds. (*P.U. 1966*)

5·4 Velocity at time t.

Let V be the velocity of the particle at time t. Let the direction of the velocity (or the direction of the tangent) make an angle θ with the horizontal. Then



$$V \cos \theta = \frac{dx}{dt} = u \cos \alpha \quad \ldots(1)$$

and
$$V \sin \theta = \frac{dy}{dt} = u \sin \alpha - gt \quad \ldots(2)$$

Dividing (2) by (1), we get

$$\tan \theta = \frac{u \sin \alpha - gt}{u \cos \alpha}$$

which gives the direction of the velocity.

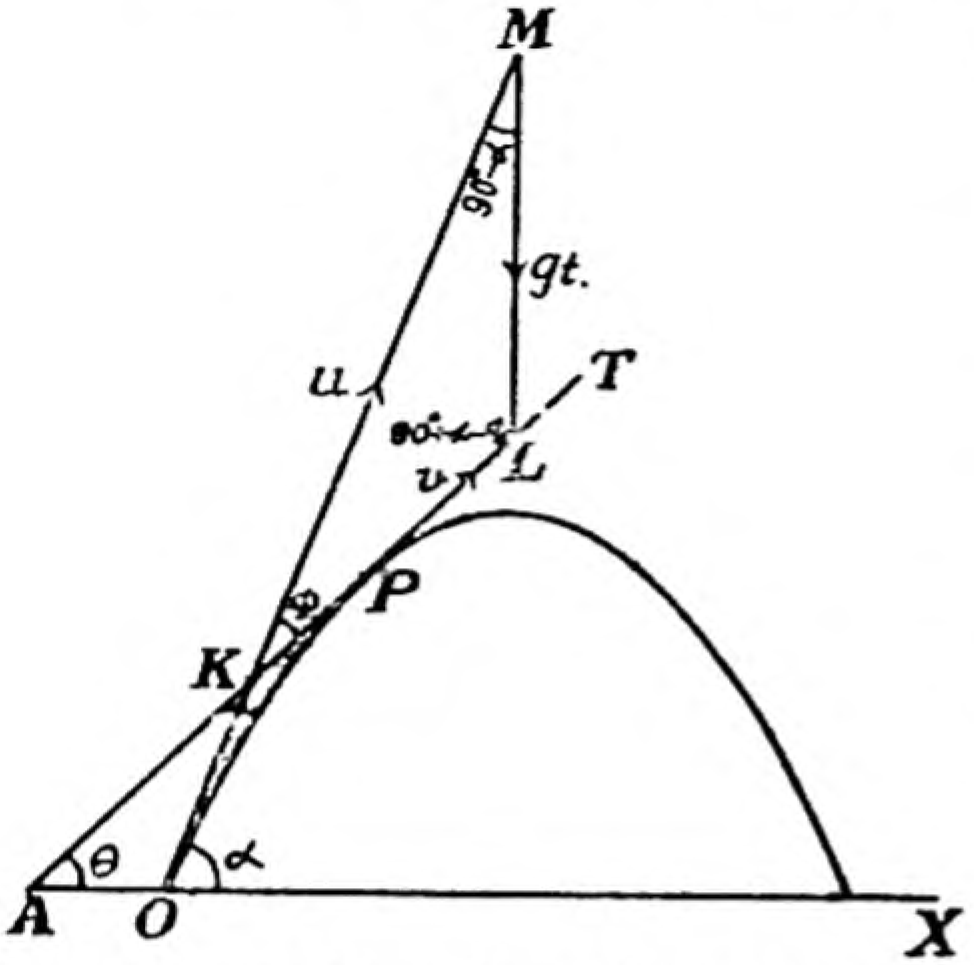
Squaring and adding (1) and (2),

$$\begin{aligned} V^2 &= u^2 \cos^2 \alpha + (u \sin \alpha - gt)^2 \\ &= u^2 - 2u \sin \alpha\, gt + g^2t^2 \\ &= u^2 - 2g\,(u \sin \alpha\, t - \tfrac{1}{2}\, gt^2) \\ &= u^2 - 2gy\ ; \ y \text{ being the ordinate of P.} \\ &= 2g\left(\frac{u^2}{2g} - y\right) \\ &= 2g\ (\text{height of the directrix} - \text{ordinate of P}) \\ &= 2g\ (\text{height of the directrix above P}) \end{aligned}$$

Hence the magnitude of the velocity at any point is the same as would be acquired in falling freely a vertical distance equal to the depth of the point below the directrix. (*P.U.*)

5·41 Change in the Direction of Motion of a Projectile

Let u be the initial velocity acting along OM ; the tangent at O. Let the velocity v, at P, along the tangent PT, meet u at K.



Let ϕ be the angle through which the direction of motion turns in time t, then $\angle \mathrm{MKP}=\phi$.

Obviously $\phi=\alpha-\theta$ ...(I)

Let u, v be represented on a certain scale by $\overline{\mathrm{KM}}$ and $\overline{\mathrm{KL}}$; then $\overline{\mathrm{ML}}$, represents, by the Triangle Law of Velocities, the Velocity which compounded with u, gives the resultant v.

Since gravity is the only force acting during the motion of a Projectile, the change produced by the acceleration due to gravity (g) in time t is gt in the vertically downward direction.

Hence $\overline{\mathrm{ML}}$ represents gt and is vertical.

Thus, $\angle \mathrm{OML}=90°-\alpha$,

so that $\angle \mathrm{KLM}=90°+\alpha-\phi$.

Since u, gt, v are represented by the sides of the $\triangle$ KML :

$$\frac{u}{\mathrm{KM}}=\frac{gt}{\mathrm{ML}}=\frac{v}{\mathrm{KL}}.$$

Whence by sine formula,

$$\frac{u}{\sin(90°+\alpha-\phi)}=\frac{gt}{\sin\phi}=\frac{v}{\sin(90°-\alpha)}$$

or

$$\frac{u}{\cos(\alpha-\phi)}=\frac{gt}{\sin\phi}=\frac{v}{\cos\alpha} \qquad \text{...(II)}$$

5·42 Results. (1) ϕ, the change in the direction of motion in t is given by

$$\sin\phi=\frac{gt\cos\alpha}{v} \qquad \text{...(III)}$$

(2) ϕ **is also given by**

$$gt\cos(\alpha-\phi)=u\sin\phi$$

or

$$gt\cos\phi\cos\alpha+gt\sin\phi\sin\alpha=u\sin\phi$$

or

$$\tan\phi=\frac{gt\cos\alpha}{u-gt\sin\alpha} \qquad \text{...(IV)}$$

(3) If $\phi=90^\circ$, *i.e.*, the velocity is perpendicular to its original direction, we get from (IV), $\tan\phi\to\infty$.

$$\therefore \qquad u-gt\sin\alpha=0 \quad \text{or} \quad t=\frac{u}{g\sin\alpha}.$$

or from (II),

$$\frac{u}{\cos(\alpha-90^\circ)}=\frac{gt}{\sin 90^\circ}$$

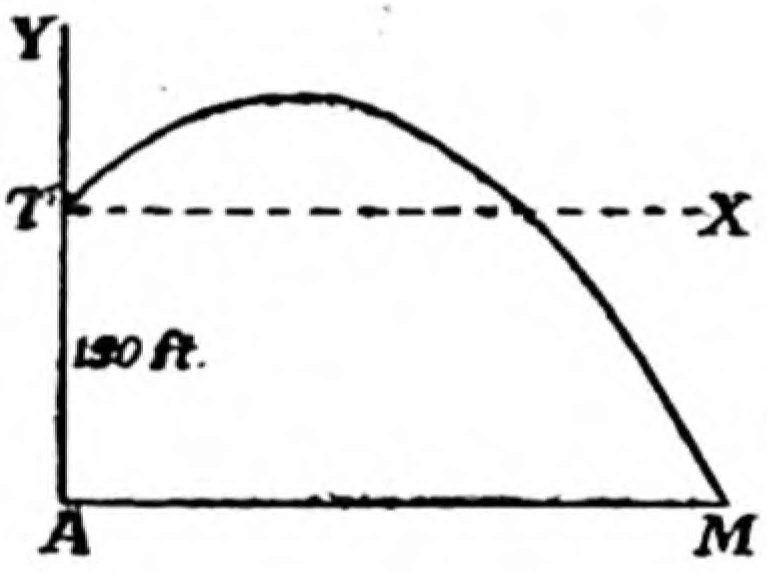
giving again,

$$t=\frac{u}{g\sin\alpha} \qquad \ldots\text{(V)}$$

Thus, after the time $\frac{u}{g\sin\alpha}$, *the projectile will be moving perpendicular to its original direction of motion.* (*P.U.*)

Ex. 7. *A stone is projected with a velocity of 100 f.p.s. at an elevation of 30° from a tower 150 ft. high. Find (a) the point when it strikes the horizontal through the foot of tower (b) the velocity on striking the ground.* (*P.U.*)

Let AT be the tower.



Taking the horizontal and the vertical lines through T the top of the tower as axes the equation of the trajectory is

$$y=x\tan 30^\circ-\frac{gx^2}{20000\cos^2 30^\circ}$$

$$y=\frac{x}{\sqrt{3}}-\frac{4x^2}{1875}. \qquad \ldots\text{(I)}$$

Equation of AM, referred to TX, TY as axes is $y=-150$. ...(II)

Solving (I) and (II) to get x, we have $-150=\frac{x}{\sqrt{3}}-\frac{4x^2}{1875}$.

Whence we get $x=-\frac{375\sqrt{3}}{4}$ or $250\sqrt{3}$ giving $AM=250\sqrt{3}$ ft.

Or thus,

$$x=u\cos\alpha . t=100\times\frac{\sqrt{3}}{2}t=50\sqrt{3}\,t$$

$$y=u\sin\alpha\, t-\tfrac{1}{2}gt^2=50t-16t^2.$$

To get t, put $y=-150$, whence $-150=50t-16t^2$

giving $t=-\frac{15}{8}, 5$; hence $t=5$ sec.

Correspondingly, $AM=x=50\sqrt{3}\times 5=250\sqrt{3}$ ft.

Further

$$\frac{dx}{dt}=u\cos\alpha=50\sqrt{3}.$$

$$\frac{dy}{dt}=u\sin\alpha-gt=50-32\times 5=-110.$$

$$\therefore \qquad V=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)}=140 \text{ f.p.s.}$$

$$\text{And} \qquad \tan\theta=\frac{-110}{50\sqrt{3}}=-\frac{11}{5\sqrt{3}}.$$

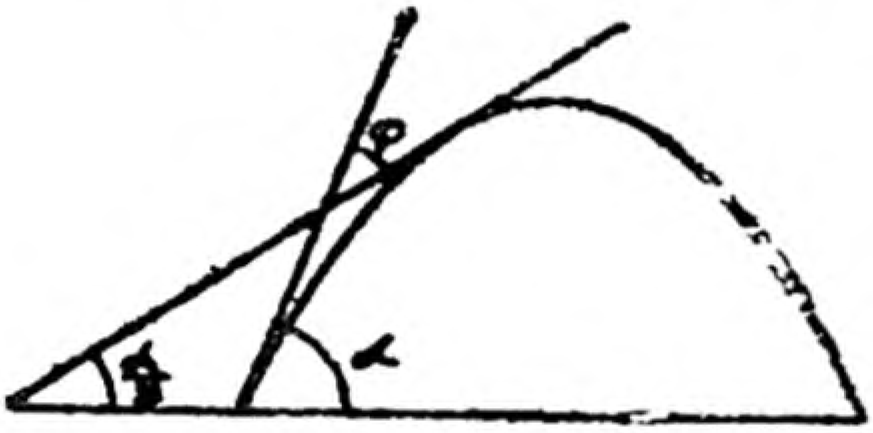
Also, we could get, $V^2=u^2-2gy=100^2-2\times 32\times(-150)$

$$=19600.$$

$$\therefore \qquad V=140 \text{ f.p.s. as before.}$$

Ex. 8. *A body is projected at an angle of elevation α with a velocity of 32 f.p.s. In what time will the direction of motion be ½α ?* *(P.U. 1949 S)*

Sol. By Result IV of Art. 5·42



$$\tan\phi=\frac{gt\cos\alpha}{u-gt\sin\alpha}$$

$$\text{Since} \qquad \phi=\alpha-a/2=\alpha/2.$$

$$\therefore \qquad \tan\frac{\alpha}{2}=\frac{32t\cos\alpha}{32-32t\sin\alpha}$$

$$=\frac{t\cos\alpha}{1-t\sin\alpha}$$

$$\text{or} \qquad \tan\frac{\alpha}{2}-t\tan\frac{\alpha}{2}\sin\alpha=t\cos\alpha$$

$$\text{giving} \qquad t=\frac{\tan\frac{\alpha}{2}}{\tan\frac{\alpha}{2}\sin\alpha+\cos\alpha}$$

$$=\frac{\tan\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\,\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}+1-2\sin^2\frac{\alpha}{2}}$$

$$=\tan\frac{\alpha}{2} \text{ seconds.}$$

Exercise V (B)

1. A particle is projected under gravity with a velocity 64 ft./sec. at an angle of elevation 30°. Find its speed (*a*) after half a second, (*b*) at the highest point of the path, (*c*) at a height of 15 ft.

Find also the interval of time between two points at which the particle has the same speed of 56 ft./sec.

2. A particle is projected from a point O in a vertical plane. If, after one second of its projection, it has a velocity of 96 f.p.s in a direction making an angle of 45° with the horizontal, find the time of flight and the range on the horizontal plane through O.

(P.U. B.Sc., 1960)

3. Two seconds after its projection, a projectile is travelling in a direction inclined at 30° to the horizon, after one more second it is travelling horizontally. Determine the magnitude and direction of its initial velocity. *(P.U. B.Sc., 1960 S)*

4. If v_1 and v_2 be the velocities at the ends of a focal chord of a projectile's path and u the horizontal component of the velocity, show that

$$\frac{1}{v_1^2}+\frac{1}{v_2^2}=\frac{1}{u^2}. \qquad \textit{(P.U. 1961 S)}$$

5. Two particles are projected simultaneously in the same vertical plane from the same point with velocities u and v at angles α and β to the horizontal. Show that

(*i*) the line joining them moves parallel to itself ;

(*ii*) the time that elapses, before their velocities are parallel, is

$$\frac{uv\sin(\alpha-\beta)}{g(v\cos\beta-u\cos\alpha)}\,; \qquad \textit{(P.U. 1960)}$$

(*iii*) the time that elapses between their transits through the other common point is

$$\frac{2uv\sin(\alpha-\beta)}{g(u\cos\alpha+v\cos\beta)}.$$

6. At any instant a projectile is moving with a velocity u in a direction making an angle α to the horizon. After a certain interval t the direction of its path makes an angle β with the horizontal. Prove that

$$u\cos\alpha=\frac{gt}{\tan\alpha-\tan\beta}.$$

Prove also that the direction of motion turns through an angle θ after a time $\dfrac{u\sin\theta}{g\cos(\theta-\alpha)}$ and that the direction of the motion will be at right angles to the former direction after a time

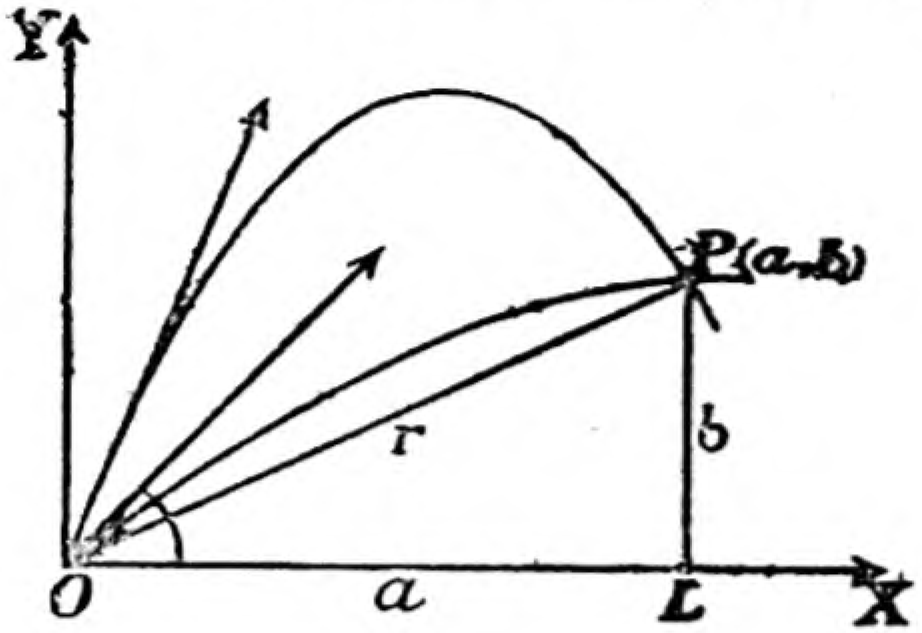
$$\frac{u}{g\sin\alpha}.$$

7. A particle is projected with velocity V and elevation $\alpha\left(<\frac{\pi}{4}\right)$ from a point O situated at a height h above a horizontal plane. If the particle is moving at right angles to its direction when it hits the plane after time t, show that

(*i*) $t=\dfrac{V}{g\sin\alpha}$,

(*ii*) $h=\dfrac{V^2\cos2\alpha}{2g\sin^2\alpha}$.

5·5 Direction of projection to hit a given point.



The equation of the path is

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \qquad \text{...(I)}$$

Let the particle be required to hit the point P(a, b).

$$\text{Then } b = a\tan\alpha - \frac{ga^2}{2u^2\cos^2\alpha}$$

$$= a\tan\alpha - \frac{ga^2}{2u^2}(1+\tan^2\alpha)$$

or
$$\tan^2\alpha - \frac{2u^2}{ga}\tan\alpha + 1 + \frac{2u^2b}{ga^2} \qquad \text{...(II)}$$

This is quadratic equation in tan α and thus gives two values of tan α. We, therefore, get two directions of projection to hit the point (a, b).

The roots of (II) are real if

$$\left(\frac{2u^2}{ga}\right)^2 \geqslant 4\left(1 + \frac{2u^2b}{ga^2}\right)$$

or
$$4u^4 \geqslant 4g(ga^2 + 2u^2b)$$

or
$$4u^4 \geqslant g^2a^2 + 2u^2gb$$

or
$$u^4 - 2u^2gb + g^2b^2 \geqslant g^2a^2 + g^2b^2$$

or
$$(u^2 - gb)^2 \geqslant g^2(a^2 + b^2)$$

or
$$u^2 - gb \geqslant g\sqrt{a^2+b^2}$$

or
$$u^2 \geqslant g(b + \sqrt{a^2+b^2})$$

Hence the least value of the velocity of projection, u to hit a given point (a, b) is given by

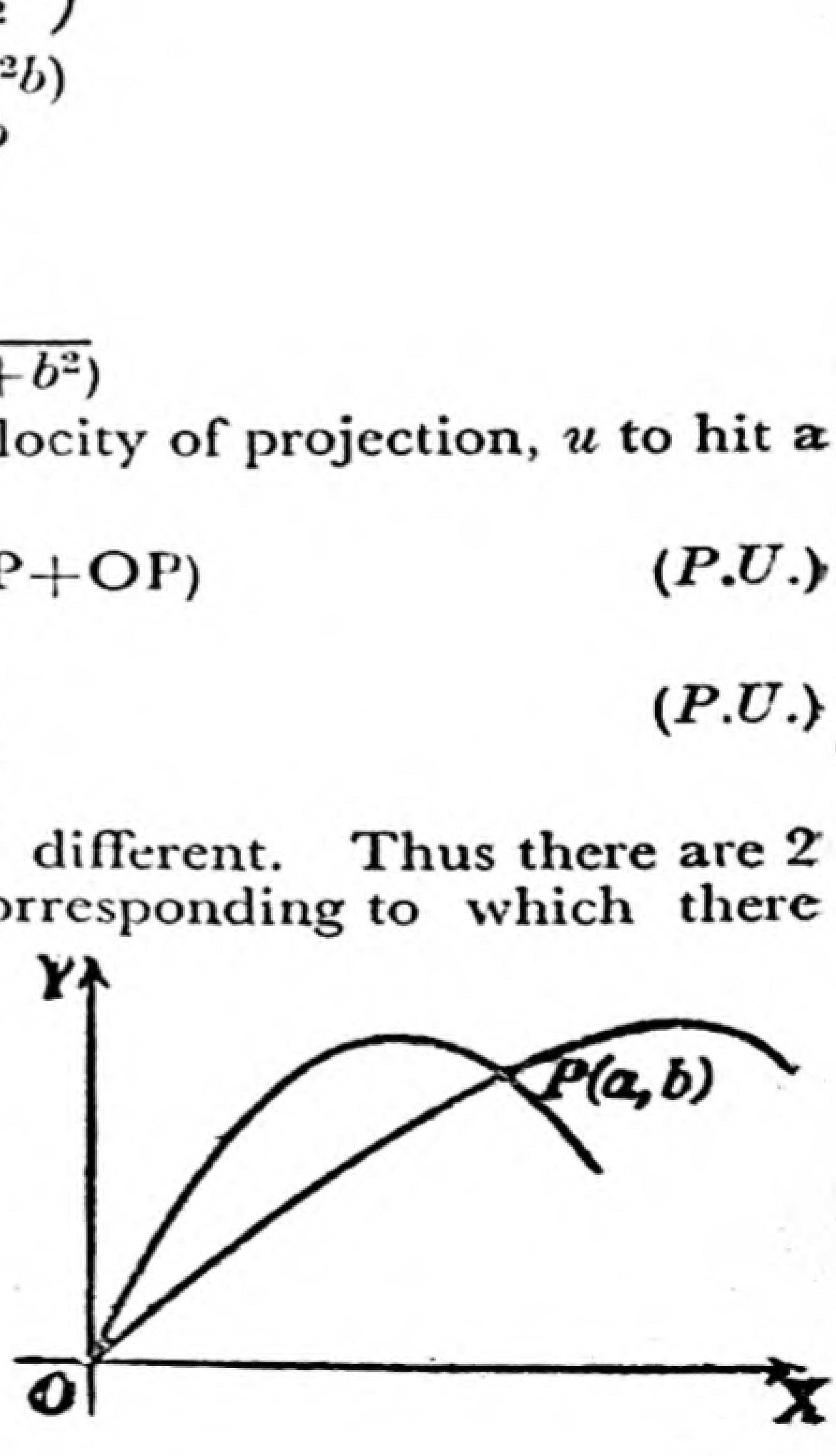
$$u^2 = g(b + \sqrt{a^2+b^2}) = g(\text{LP} + \text{OP}) \qquad (P.U.)$$

5·51 *To discuss the case when*

$$u^2 \gtreqless g\{b + \sqrt{a^2+b^2}\}. \qquad (P.U.)$$

When
$$u^2 > g[b + \sqrt{a^2+b^2}]$$

the roots of (II) (Art. 5·5) are real and different. Thus there are 2 distinct values of tan α and hence of α, corresponding to which there are 2 paths of the projectile to pass through P(a, b). In one case it passes through P and then rises to the maximum height, while in the other it rises to the maximum height first and then passes through P. The two paths coincide when $u^2 = g[b + \sqrt{a^2+b^2}]$. If however, $u^2 < g[b + \sqrt{a^2+b^2}]$ the roots of (II) are imaginary that is, the particle cannot pass through the point P(a, b) if



$$u^2 < g(b + \sqrt{a^2+b^2}).$$

5·52 Times to pass through a given point.

The position of the particle after time t is give by the equations

$$x=u\cos\alpha t$$
$$y=u\sin\alpha t-\tfrac{1}{2}gt^2.$$

If the particle passes through the point (a, b),

$$a=u\cos\alpha t \qquad \ldots(1)$$
$$b=u\sin\alpha-\tfrac{1}{2}gt^2.$$

The second equation can be written as

$$b+\tfrac{1}{2}gt^2=u\sin\alpha t \qquad \ldots(2)$$

Squaring and adding (1) and (2),

$$a^2+(b+\tfrac{1}{2}gt^2)^2=u^2t^2$$

or

$$g^2t^4+4(gb-u^2)t^2+4(a^2+b^2)=0.$$

This is a quadratic equation in t^2 and gives two roots. Let t_1^2 and t_2^2 be its roots.

$$\text{Product of the roots}=t_1^2t_2^2=\frac{4(a^2+b^2)}{g^2}$$
$$=\frac{4r^2}{g^2}\ ;$$

r being the distance of the point (a, b) from the point of projection

$$\therefore \qquad t_1t_2=\frac{2r}{g}=\frac{2\text{OP}}{g} \qquad \ldots(3) \qquad (D.U.\ ;\ P.U.)$$

which shows that the product of the two times of transits through the point is independent of the velocity of projection.

Now the time, t, taken by a particle in falling vertically from rest through a distance r is given by

$$r=\tfrac{1}{2}gt^2$$

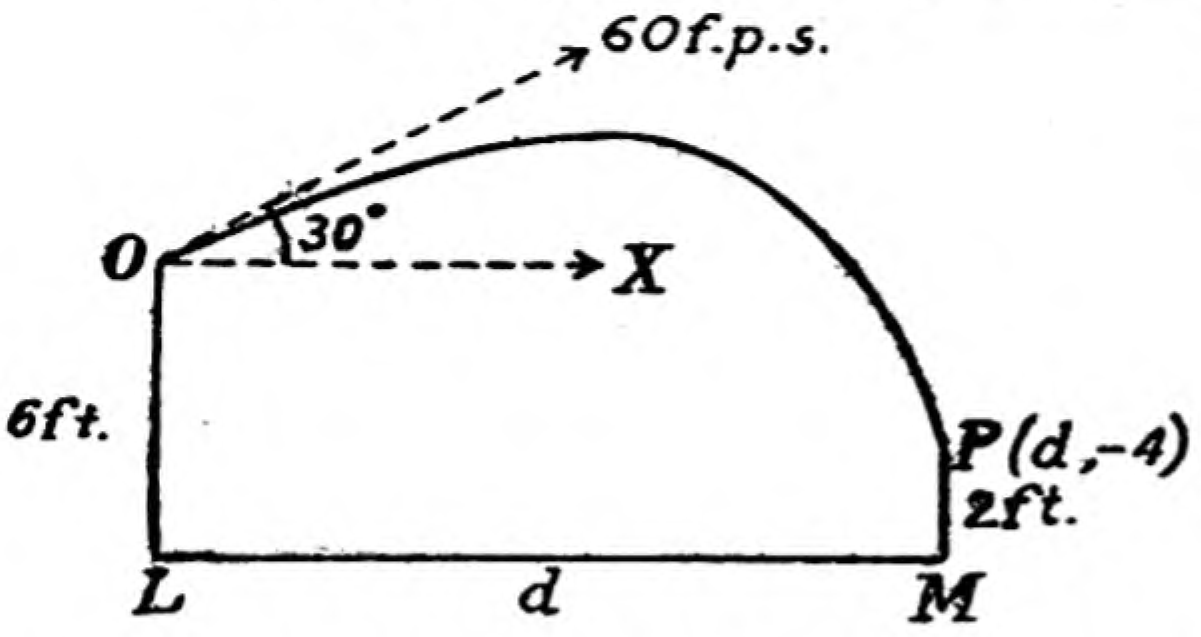
or

$$t^2=\frac{2r}{g}=\frac{2\text{OP}}{g} \qquad \ldots(4)$$

From (3) and (4) it follows that *the product of the two times of transits of a projectile through a given point is equal to the square of the time taken to fall from rest through a vertical height $r(=OP)$.*

Ex. 9. *A cricket ball, thrown from a height of 6 ft. at an angle of 30° with the horizon, with a speed of 60 f.p.s. is caught by another fieldman at a height of 2 ft. from the ground. How far apart were the two men?* (P.U.)

Taking the point of projection O, as origin and the horizontal and vertical through O as axes ; the equation of the path of ball is



$$v=x\tan 30^\circ-\frac{gx^2}{2\times 60^2\times \cos^2 30^\circ}$$

or $$y=\frac{x}{\sqrt{3}}-\frac{4x^2}{675}. \quad \text{...(I)}$$

Let the point P where the ball is caught be at a distance of d from O, then $P\equiv(d, -4)$.

$\because$ P is on the path (I)

$\therefore$ $-4=\frac{d}{\sqrt{3}}-\frac{4d^2}{675}$, giving $d=60\sqrt{3}$ ft.

Or thus :—

$$x=u\cos\alpha.t=30\sqrt{3}\,t$$
$$y=u\sin\alpha.t-\tfrac{1}{2}gt^2=30t-16t^2.$$

Where the ball is caught, $y=-4$.

$\therefore$ $-4=30t-16t^2$, giving $t=2$ sec.

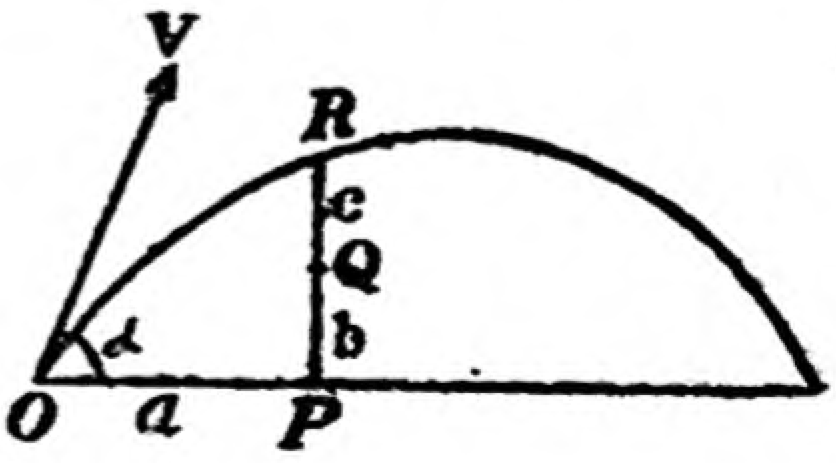
whence $x=30\sqrt{3}\times 2=60\sqrt{3}$ feet.

Ex. 10. *A ball is projected from a point on the ground distant a from the foot of a vertical wall of height b, the velocity of projection being V at an angle α to the horizontal. Find how high above the wall the ball passes it. If the ball just clears the wall, prove that the greatest height reached is*

$$\frac{a^2\tan^2\alpha}{4(a\tan\alpha-b)}.$$ *(Kharagpur, 1958)*

Let PQ be the wall. Let the ball pass at a height c above Q so that $QR=c$.

The equation of the path is



$$y=x\tan\alpha-\tfrac{1}{2}\frac{gx^2}{V^2\cos^2\alpha}.$$

The point R $(a, b+c)$ lies on it.

$$\therefore \quad b+c=a\tan\alpha-\tfrac{1}{2}\frac{ga^2}{V^2\cos^2\alpha}.$$

$$\therefore \quad c=a\tan\alpha-\tfrac{1}{2}\frac{ga^2}{V^2\cos^2\alpha}-b.$$

If the ball just clears the wall, $c=0$

$$\therefore \quad a\tan\alpha-\tfrac{1}{2}\frac{ga^2}{V^2\cos^2\alpha}-b=0.$$

or
$$a \tan \alpha - b = \tfrac{1}{2}\frac{ga^2}{V^2 \cos^2 \alpha}$$

or
$$V^2 = \frac{ga^2}{2 \cos^2 \alpha(a \tan \alpha - b)}.$$

The greatest height reached$=\dfrac{V^2 \sin^2 \alpha}{2g}$

$$= \frac{\sin^2 a}{2g} \cdot \frac{ga^2}{2 \cos^2 \alpha(a \tan \alpha - b)}$$

$$= \frac{a^2 \tan^2 \alpha}{4(a \tan \alpha - b)}.$$

Ex. 11. *A body is projected at an angle α to the horizontal, so as to clear two walls of equal height a at a distance 2a from each other. Show that the range is equal to 2a cot ½ α.* *(P.U. 1955 S)*

The equation of the path is
$$y = x \tan \alpha - \frac{1}{2}\,\frac{gx^2}{u^2 \cos^2 \alpha}.$$

It clears the wall of height a at a distance x

∴
$$a = x \tan \alpha = \frac{1}{2}\,\frac{gx^2}{u^2 \cos^2 \alpha}$$

or
$$gx^2 - 2u^2 \sin \alpha \cos \alpha + 2au^2 \cos^2 a = 0$$

which is a quadratic equation in x. Let x_1 and x_2 be its roots. Then

$$x_1 + x_2 = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \text{R, the horizontal range}$$

∴
$$u^2 = \frac{g\text{R}}{2 \sin \alpha \cos a}$$

Also
$$x_1 x_2 = \frac{2au^2 \cos^2 \alpha}{g}.$$

The distance between the walls is
$$2a = x_2 - x_1$$

or
$$4a^2 = (x_2 + x_1)^2 - 4x_1 x_2$$
$$= \text{R}^2 - \frac{8au^2 \cos^2 \alpha}{g}$$
$$= \text{R}^2 - \frac{8a \cos^2 \alpha}{g} \cdot \frac{g\text{R}}{2 \sin \alpha \cos \alpha}$$

or
$$\text{R}^2 - 4a \cot \alpha \,.\, \text{R} - 4a^2 = 0$$

∴
$$\text{R} = \frac{4a \cot \alpha + \sqrt{16a^2 \cot^2 \alpha + 16a^2}}{2}, \text{ rejecting the negative root}$$

$$= 2a \cot \alpha + 2a \operatorname{cosec} \alpha$$

$$= 2a\, \frac{\cos \alpha + 1}{\sin \alpha}$$

$$=2a\,.\,\frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$=2a\cot\frac{\alpha}{2}.$$

Exercise V (C)

1. A body, thrown in a direction making an angle of 30°, with the horizon passes through a point $400\sqrt{3}$ ft. horizontally from the point of projection and 76 ft. above it. Find the velocity of projection. *(P.U.)*

2. A particle is projected from a point A with a speed of 8 f.p.s. Determine the directions of projection for which it will pass through a point whose height above A is 6 inches and whose horizontal distance from A is one foot. *(P.U.)*

3. A jet of water from a nozzle at ground level clears a wall 12 ft. high and at horizontal distance 18 ft. from the nozzle and also the top of a house 32 ft. high and at 75 ft. horizontal distance from the nozzle. Find V and θ. *(P.U.)*

4. A particle is required to be projected with a given velocity and a certain angle of projection so that it just passes over the wall of a fort horizontally. If the wall be 64 ft. high and 192 ft. distant from the gun, find the angle and velocity of projection. *(P.U.)*

5. From a point on the ground at a distance of x from the foot of a wall, a ball is thrown at an angle of 45°, which just clears the top of the wall and afterwards strikes the ground at a distance y on the other side. Find the height of the wall. *(P.U.)*

6. A regular hexagon stands with one side on the ground and a particle is projected so as to graze its four upper vertices. Show that the velocity on reaching the ground is to its least velocity as $\sqrt{31} : \sqrt{3}$.

7. From the ground, a stone is rightly aimed and thrown at a crow sitting on a wall $31\frac{1}{4}$ ft. high. Just before being hit by the stone, the crow flies away horizontally with a velocity of 24 ft. per sec. Notwithstanding this, the stone, after rising 16 ft. higher, does hit him in its downward motion. Find the initial velocity of the stone. *(P.U.)*

8. How must a ball be projected from a height of 4 ft. so as just to clear a wall 13 ft. high distant 15 ft. in a horizontal direction and a ditch 5 ft. wide on the other side of the wall. *(P.U. 1964)*

9. A particle is projected with velocity $2\sqrt{ag}$ so that it just clears two walls of equal height a, which are at a distance $2a$ from each other. Show that the latus rectum of the path is $2a$ and that the time of passing between the walls is

$$2\sqrt{\frac{a}{g}}.$$

(*Delhi, 1957 ; Panjab, '56 ; Raj., '53*)

10. A particle is thrown over an isosceles right-angled triangle ABC, right-angled at B, from one end A of the horizontal base AC and grazing the vertex B falls at C. Show that the angle of projection is $\tan^{-1} 2$. (*P.U. 1968 S*)

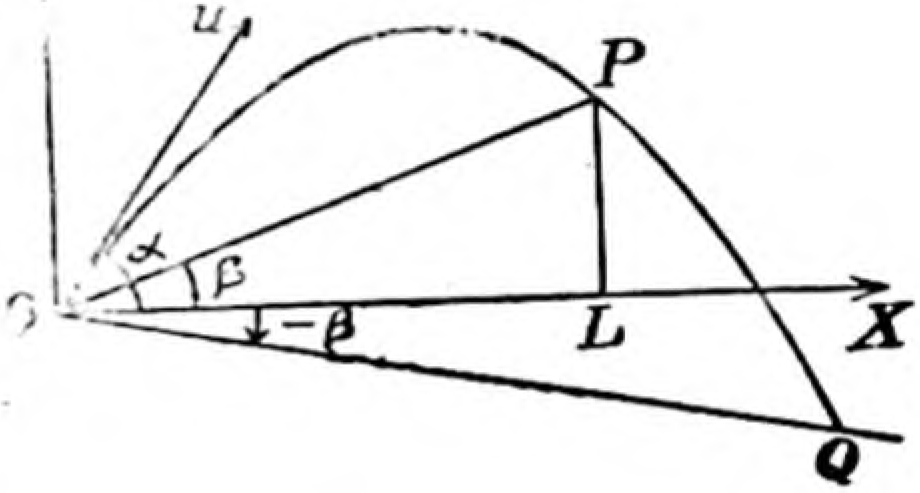
11. A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If α and β are the base angles and θ the angle of projection, prove that

$$\tan\theta = \tan\alpha + \tan\beta.$$

(*Delhi, 1962 ; Roorkee, '59 ; Panjab, '67*)

5·6 Range on an inclined plane.

Let a particle be projected with velocity u at elevation α from a point O on a plane of inclination β on the line of greatest slope. Let the particle strike the inclined plane at P.



Then OP is the range on the inclined plane. Let $OP = r$.

Taking O as the origin and the horizontal and vertical lines through O as the axes of x and y respectively, the co-ordinates of P are $(r\cos\beta, r\sin\beta)$.

The equation of the path is

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$$

$$\therefore \qquad r\sin\beta = r\cos\beta\tan\alpha - \frac{gr^2\cos^2\beta}{2u^2\cos^2\alpha}$$

or

$$\frac{\frac{1}{2}gr\cos^2\beta}{u^2\cos^2\alpha} = \cos\beta\tan\alpha - \sin\beta$$

$$= \frac{\sin(\alpha-\beta)}{\cos\alpha}$$

$$\therefore \qquad r = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

$$= \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta].$$

5·61 If the particle is projected **down the plane** then changing β to $-\beta$, we get OQ, range down the plane

$$= \frac{u^2}{g\cos^2\beta}[\sin(2a+\beta)+\sin\beta].$$

5·62 Maximum range. u and β being given, the range r varies with α.

$\therefore$ r is maximum when $\sin(2\alpha-\beta)$ is maximum.

i.e. when $\quad \sin(2\alpha-\beta)=1$

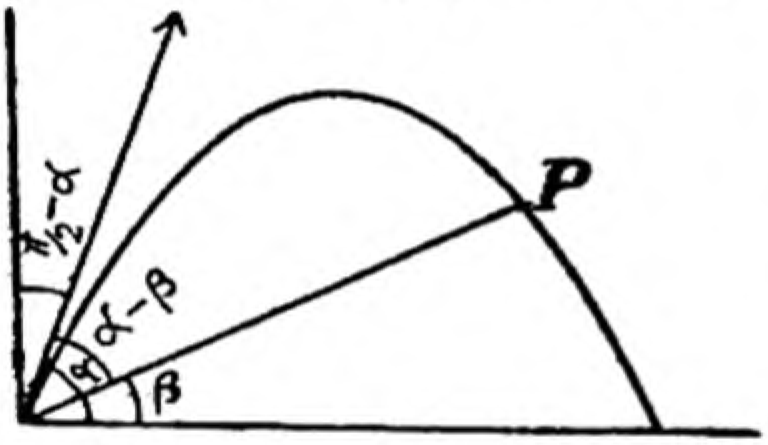
or when $$2\alpha-\beta=\frac{\pi}{2} \qquad \ldots(A)$$

or when $$\alpha=\frac{\pi}{4}+\frac{\beta}{2}$$

This gives the direction of projection for the maximum range on the inclined plane and this maximum range

$$=\frac{u^2(1-\sin\beta)}{g\cos^2\beta}$$
$$=\frac{u^2}{g(1+\sin\beta)}$$

5·63 (A) can also be written as



$$\alpha-\beta=\frac{\pi}{2}-\alpha.$$

Hence for the maximum range, the direction of projection must bisect the angle between the vertical and the inclined plane.

5·64 Maximum range down the plane

$$=\frac{\mathbf{u^2}}{\mathbf{g(1-\sin\beta)}}.$$

5·65 Direction of projection for a given range.

Range up the inclined plane is

$$r=\frac{u^2}{g\cos^2\beta}\left[\sin(2\alpha-\beta)-\sin\beta\right].$$

Now $$\sin(2\alpha-\beta)=\sin\left[\pi-(2\alpha-\beta)\right]$$
$$=\sin\left[\pi+2\beta-2\alpha)-\beta\right]$$
$$=\sin\left[2\left(\frac{\pi}{2}+\beta-\alpha\right)-\beta\right]$$
$$=\sin(2\alpha_1-\beta),$$

where $$\alpha_1=\frac{\pi}{2}+\beta-\alpha.$$

$\therefore$ *the directions* α *and* $\alpha_1\left(=\frac{\pi}{2}+\beta-\alpha\right)$ *give the same range.*

Again $$\alpha-\beta=\frac{\pi}{2}-\left(\frac{\pi}{2}+\beta-\alpha\right)$$
$$=\frac{\pi}{2}-\alpha_1 \qquad \ldots(1)$$

$\therefore$ the angle between one direction of projection and the inclined plane=angle between the other direction of projection and the vertical.

From (1),

$$\alpha-\frac{\beta}{2}-\frac{\pi}{4}=\frac{\pi}{4}+\frac{\beta}{2}-\alpha_1,$$

or

$$\alpha-\left(\frac{\pi}{4}+\frac{\beta}{2}\right)=\left(\frac{\pi}{4}+\frac{\beta}{2}\right)-\alpha_1,$$

$\frac{\pi}{4}+\frac{\beta}{2}$ being the direction for the maximum range, it follows that for a given range the two directions of projection are such that they are equally inclined to the direction for the maximum range.

5·66 Time of Flight.

Let the particle hit the plane after time T.

Then OL=horizontal distance described by the particle in time T

$$=u\cos\alpha\,.\,\mathrm{T}$$

Also $\mathrm{OL}=r\cos\beta$

$$\therefore\quad r\cos\beta=u\cos\alpha\,.\,\mathrm{T}$$

$$\therefore\quad \mathrm{T}=\frac{r\cos\beta}{u\cos\alpha},\text{ where } r=\frac{2u^2}{g}\,\frac{\sin(\alpha-\beta)\cos\alpha}{\cos^2\beta}$$

$$=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}.$$

5·67 Velocity at P.

Let v be the velocity of the particle when it hits the plane.

Then $v^2=u^2-2gy$, where $y=\mathrm{PL}$

$$=u^2-2gr\sin\beta.$$

Let θ be the angle which the direction at P makes with the horizontal. Then

$$\tan\theta=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{u\sin\alpha-gt}{u\cos\alpha}$$

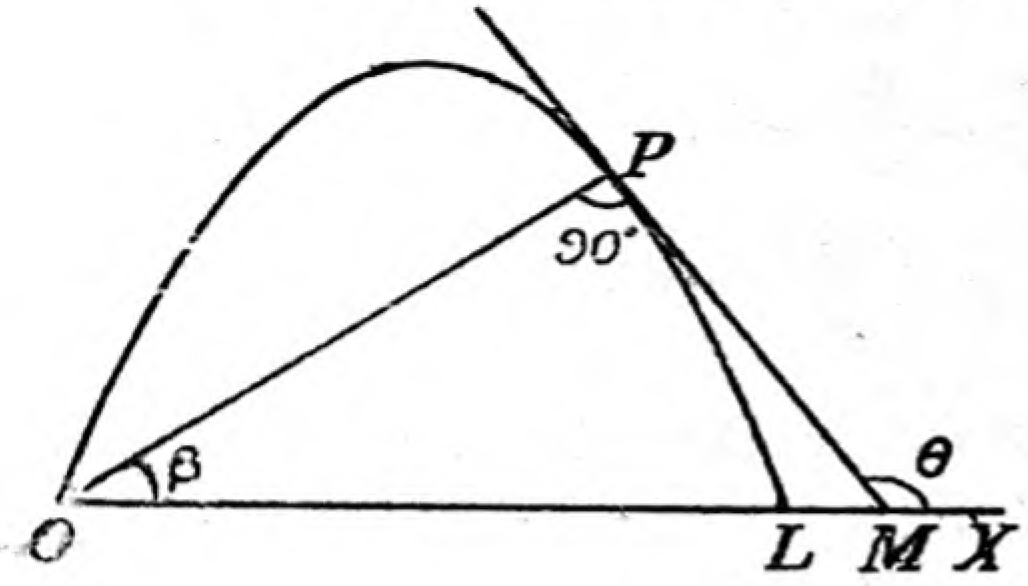
$$=\tan\alpha-\frac{g}{u\cos\alpha}\cdot\frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

$$=\tan\alpha-2(\tan\alpha-\tan\beta)$$

$$=2\tan\beta-\tan\alpha.$$

5·68 *Condition that the particle may strike the plane at right angles.*

If the particle strikes the plane at right angles then



$$\theta=\frac{\pi}{2}+\beta.$$

$$\therefore\ \tan\left(\frac{\pi}{2}+\beta\right)=2\tan\beta-\tan\alpha$$

or $$-\cot\beta=2\tan\beta-\tan\alpha$$

or $$\cot\beta=\tan\alpha-2\tan\beta\cdot$$

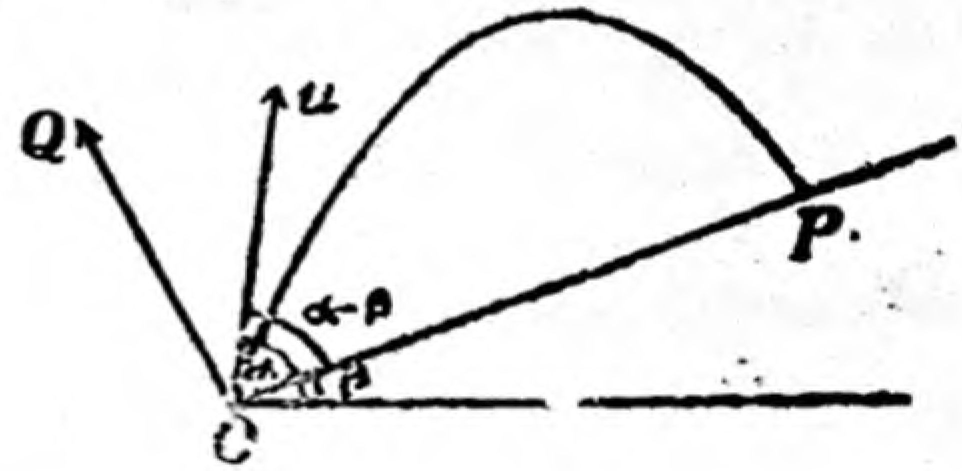
By adding $\tan\alpha$ to both sides and simplifying, the required condition can be put in the form

$$\cot\beta=2\tan(\alpha-\beta).$$

5·7 Alternative Method.

Let OQ be a line in the vertical plane perpendicular to OP.

Then



component of u along OP
$=u\cos(\alpha-\beta)$
component of u along OQ
$=u\sin(\alpha-\beta)$
component of g along OP
$=-g\sin\beta$
component of g along OQ
$=-g\cos\beta.$

5 71 To find the *time af flight*, we consider the motion perpendicular to OP.

When the particle strikes the plane, its distance perpendicular to the plane is zero.

$$\therefore\qquad 0=u\sin(\alpha-\beta)\,.\,T-\tfrac{1}{2}g\cos\beta\,.\,T^2$$

$$\therefore\qquad T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}.$$

5 72 To find the *range on the inclined plane*, we consider the motion along OP. If r be the range and T the time of flight, then

$$\begin{aligned}r&=u\cos(\alpha-\beta)T-\tfrac{1}{2}g\sin\beta.T^2\\&=u\cos(\alpha-\beta)\,.\,\frac{2u\sin(\alpha-\beta)}{g\cos\beta}-\frac{1}{2}g\sin\beta\,.\,\frac{4u^2\sin^2(\alpha-\beta)}{g^2\cos^2\beta}\\&=\frac{2u^2\sin(\alpha-\beta)}{g\cos^2\beta}[\cos(\alpha-\beta)\cos\beta-\sin(\alpha-\beta)\sin\beta]\\&=\frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta},\text{ as before.}\end{aligned}$$

5·73 *Condition that the particle may strike the plane at right angles,* we consider the motion along the plane. In this case the velocity along the plane at time T is zero.

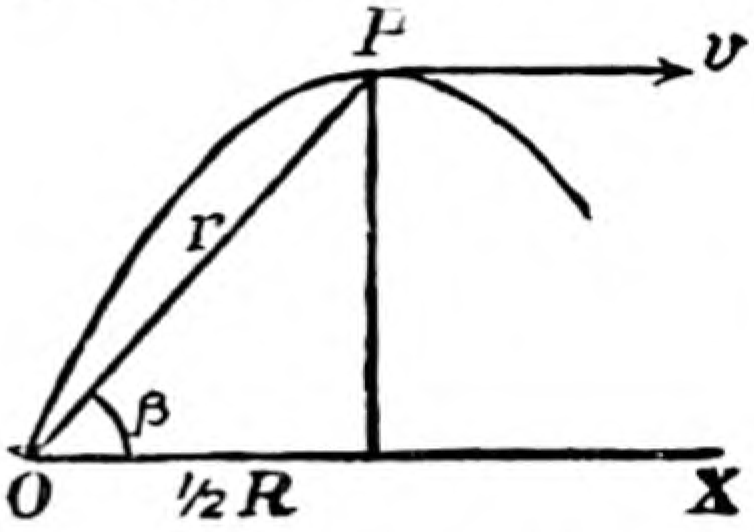
$$\therefore \qquad 0=u\cos(\alpha-\beta)-g\sin\beta.T$$

or
$$u\cos(\alpha-\beta)=g\sin\beta\,\frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

or
$$\cot\beta=2\tan(\alpha-\beta),\text{ as before.}$$

5·74 *Condition that the particle may strike the plane horizontally.*

In this case, we observe that P is at the highest point, *i.e.* the vertex ; so that



$$r\cos\beta=\frac{u^2\sin\alpha\cos\alpha}{g}$$

or
$$\frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}\cos\beta=\frac{u^2\sin\alpha\cos\alpha}{g}$$

or
$$2\sin(\alpha-\beta)=\sin\alpha\,.\,\cos\beta$$

or
$$\sin\alpha\,.\,\cos\beta=2\cos\alpha\sin\beta$$

or
$$\tan\alpha=2\tan\beta.$$

Ex. 12. *If u be the velocity of projection and v_1 the velocity of striking the plane when projected so that range up the plane is maximum, and v_2 the velocity of striking the plane when projected so that the range down the plane is maximum. Prove that*

(*P.U. 1968*)

$$v^2+v_1v_2.$$

Let β be the inclination of the plane

$$\therefore \quad \text{Maximum range up the plane}=\frac{u^2}{g(1+\sin\beta)}.$$

Height of point of striking the plane

$$=\frac{u^2}{g(1+\sin\beta)}\,.\,\sin\beta=h_1.$$

$$v_1{}^2=u^2-2gh_1$$

$$=u^2-2g\,\frac{u^2\sin\beta}{g(1+\sin\beta)}$$

$$=\frac{1-\sin\beta}{1+\sin\beta}\,u^2.$$

Similarly

$$v_2{}^2=\frac{1+\sin\beta}{1-\sin\beta}\,u$$

$$\therefore \qquad v_1{}^2v_2{}^2=u^4$$

or
$$v_1v_2=u^2.$$

Exercise V (D)

1. The greatest range, with a given velocity of projection, on a horizontal plane is 300 metres. Find the greatest range up and down a plane inclined at 30° to the horizon.

2. A particle is projected with a velocity of 9000 ft./sec. at an elevation of 60° to the horizontal from the foot of a plane of inclination 30°. Find the range on the inclined plane and the time of flight.

3. Show that for a given velocity of projection, the maximum range down an inclined plane of inclination α is greater than that up the plane in the ratio

$$\frac{1+\sin\alpha}{1-\sin\alpha}. \qquad (P.U.)$$

4. If R is the maximum range on an inclined plane through the point of projection of a particle, and T the corresponding time of flight, show that

$$R=\frac{1}{2}gT^2. \qquad (Delhi,\ 1952)$$

5. A particle is projected with speed u so as to strike at right angles a plane through the point of projection inclined at 30° to the horizon. Show that the range on the inclined plane is

$$\frac{4u^2}{7g}.$$

6. The angular elevation of an enemy's position on a hill h ft. high is β. Show that in order to shell it the initial velocity of the projectile must not be less than

$$\sqrt{gh(1+\operatorname{cosec}\beta)}.$$

7. A particle is projected from a point in a vertical plane under the action of gravity. Show that the particle strikes an inclined plane of inclination β to the horizontal at a distance

$$\frac{2u(v\cos\beta-u\sin\beta)\sec^2\beta}{g},$$

where u, v are the horizontal and vertical components of the velocity of projection. (*P.U.*)

8. A particle is projected with a velocity of 100 ft. per sec. from the foot of a plane in a direction making an angle of 45° with the plane. Find the inclination of the plane if the particle strikes it at right angles.

9. A particle projected with a velocity u from a point on an inclined plane whose inclination to the horizon is α, strikes it at right angles. Show that the range of the particle on the inclined plane is

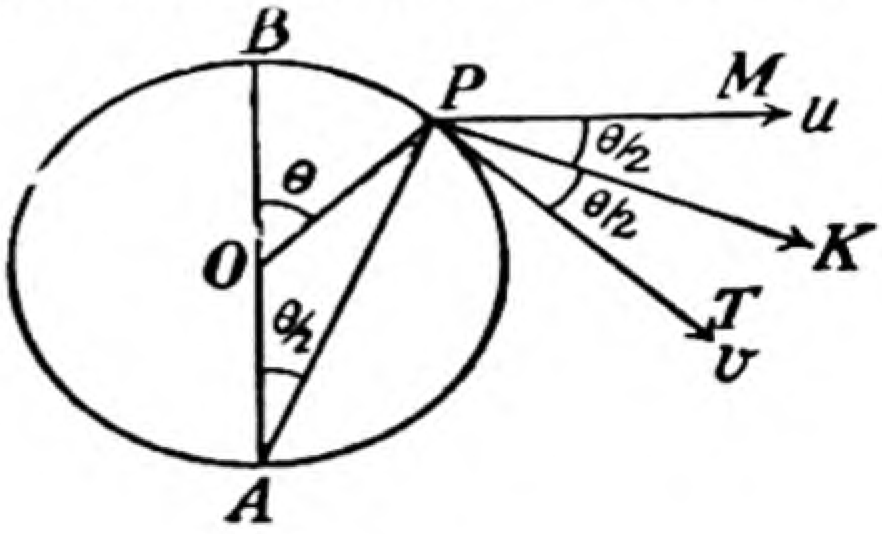
$$\frac{2u^2}{g}\cdot\frac{\sin\alpha}{1+3\sin^2\alpha}.$$

10. A shot is fired at an angle α to the horizontal up a hill of inclination β to the horizontal. Show that it strikes the hill

(*a*) horizontally if $\tan\alpha = 2\tan\beta$.

(*b*) normally if $\tan\alpha = 2\tan\beta + \cot\beta$. (*P.U., 1963*)

5·8 An Important Note. *A wheel is rolling, without slipping, in a vertical plane, along a horizontal straight line, its centre having the speed α, to find the velocity of any point on its rim.*



Let O be the centre, r the radius, A the lowest point in contact with the ground and B the highest point of the wheel. Let P be any point on the rim of the wheel.

There being no slipping, each point of the rim touches the ground in succession and thus the time taken by the point P to describe the perimeter of the wheel is the same as the time taken by the centre of the wheel to move a distance equal to the parameter.

$\therefore$ the velocity at any point P on the rim (relative to the centre) is equal to the velocity of the centre, *i.e.*, u. Hence any point P on the rim possesses two velocities each equal to u, one along the tangent at P (relative to the centre), and the other along the horizontal line PM, equal to the velocity of the centre.

If $\angle POB = \theta$, then $\angle MPT = \theta$.

$\therefore$ resultant of these two equal velocities u inclined at an angle θ

$$= 2u\cos\frac{\theta}{2} \qquad \ldots(1)$$

(along PK, where PK bisects $\angle$MPT)

$$= 2u\cos\angle PAB$$

$$= 2u\,\frac{AP}{AB}.$$

Obviously, PK is perpendicular to AP and if produced backwards it will pass through B.

Two results :

1. At B, $\theta = 0$

$\therefore$ from (1), velocity of B $= 2u$

i.e., the highest point of the wheel moves with double the velocity of the centre.

2. At A, $\theta=\pi$

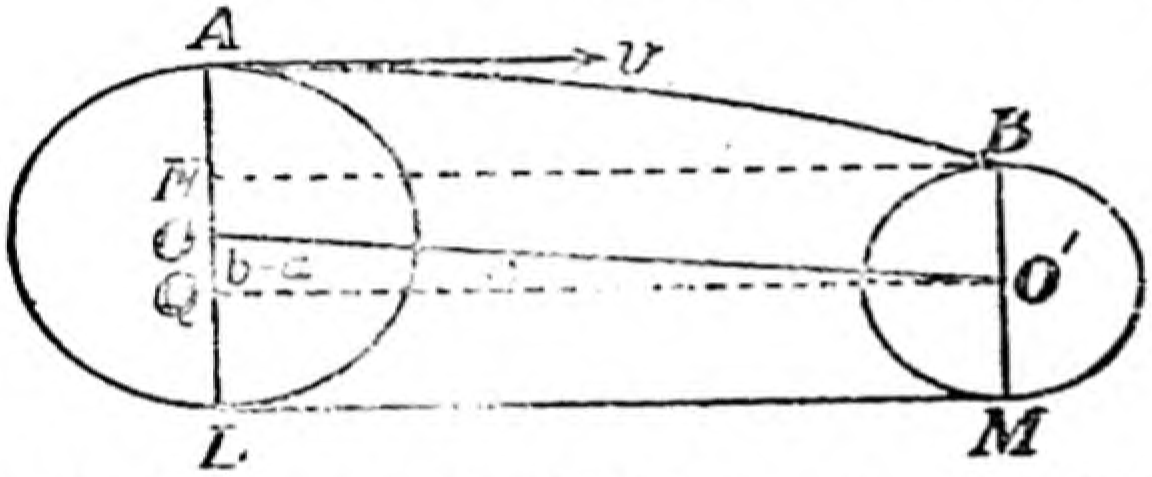
$\therefore$ from (1), velocity of A$=0$

i.e., the lowest point is instantaneously at rest.

Ex. 5. *The radii of the front and hind wheels of a carriage are a and b and c is the distance between the axle-trees. A particle of mud driven from the highest point of the hind wheel is observed to alight on the highest point of the front wheel. Show that the velocity of the carriage is*

$$\sqrt{\left(g\ \frac{(c+b-a)(c+a-b)}{4(b-a)}\right)}.$$

Let v be the velocity of the carriage so that the velocity of the highest point of the hind wheel is $2v$ horizontally.



Since the velocity of the carriage is v, the velocity of the mud particle relative to the carriage is v horizontally.

When the mud falls on the highest point of the first wheel, the vertical distance travelled by the particle

$$=\text{AL}-\text{BM}$$
$$=2b-2a$$
$$=2(b-a).$$

The horizontal distance moved by the mud

$$=\text{O}'\text{Q}$$
$$=\sqrt{c^2-(b-a)^2}.$$

Let us consider the vertical motion of the particle. If t be the time of flight,

$$2(b-a)=\tfrac{1}{2}gt^2$$

i.e.,
$$t=2\sqrt{\frac{b-a}{g}}.$$

And for the horizontal motion,

$$\sqrt{c^2-(b-a)^2}=vt$$

i.e., $\sqrt{[(c-b+a)(c+b-a)]}=v.\,2\sqrt{\dfrac{b-a}{g}}$

which gives
$$v=\sqrt{\left(g\ \frac{(c+b-a)(c+a-b)}{4(b+a)}\right)}.$$

Miscellaneous Exercise V

1. (*i*) Show that there are two directions in which a particle may be projected with a given velocity u so as to have a given horizontal range.

(*ii*) If t_1, t_2 be the times of flights for the two paths, show that

$$t_1^2+t_2^2=\frac{4u^2}{g^2}. \qquad \text{[Art. 5·32 (5)]}$$

(*iii*) Show also that if the velocity of projection be increased, the range being unaltered, the shorter time of flight decreases and the longer time of flight increases. (*P.U., 1961*)

2. A particle is projected at an elevation α and after t seconds it appears to have an elevation β as seen from the point of projection. Prove that the initial velocity was

$$\frac{gt \cos \beta}{\sin (\alpha-\beta)}.$$

(*Vikram, 1959 ; Agra, 1958, 54*)

3. A particle is projected with velocity u at an elevation α. Prove that if $h<\frac{u^2 \sin^2 \alpha}{2g}$, the particle will be again at a height h after a time

$$\frac{2}{g} \sqrt{u^2 \sin^2 \alpha-2gh}$$

after being at the same height the first time.

4. If α be the angle between the tangent at the extremities of any arc of a parabolic path, v, v_1 the velocities at these extremities and u their horizontal component, show that the time of describing the arc is

$$\frac{v\, v_1 \sin \alpha}{gu}.$$

5. A cannon fires its projectile with such an initial velocity and at such an angle of elevation that the range r and the maximum height to which the particle rises is h Find the maximum range which can be obtained with the same initial velocity.

(*P.U. B.Sc. 1961 S*)

6. A particle is projected at an angle α with the horizontal from the foot of a plane whose inclination to the horizontal is β. Show that it will strike the plane at right angles if

$$\cot \beta=2 \tan (\alpha-\beta). \qquad (\textit{Delhi, 1956})$$

7. A vertical post subtends an angle α at a point A in the same horizontal plane as the foot of the post. Two particles are projected at the same instant from A, in directions making angles θ_1 and θ_2 with the horizontal, so that the former strikes the top of the post at the same moment that the latter strikes the bottom of the post. Prove that

$$\tan \theta_1-\tan \theta_2=\tan \alpha.$$

8. A shot projected with velocity v can just reach a certain point on the horizontal plane through the point of projection. Show that, in order to hit a mark h feet above the ground at the same point, if the shot is projected at the same elevation, the velocity of projection must be increased to

$$\frac{v^2}{(v^2-gh)^{1/2}}.$$

9. A ball is projected so as just to clear two walls, the first of height a at distance b from the point of projection and the second of height b at a distance a from the point of projection. Show that the range on a horizontal plane is

$$\frac{a^2+ab+b^2}{a+b}$$

and that the angle of projection exceeds $\tan^{-1} 3$.

10. Two particles are projected simultaneously with the same speed V in the same vertical plane, but at different inclinations θ_1, θ_2. Prove that their velocities are parallel after a time

$$\frac{V}{g} \cdot \frac{\cos\frac{\theta_1-\theta_2}{2}}{\sin\frac{\theta_1+\theta_2}{2}}.$$

11. Shots fired simultaneously from the top and bottom of a vertical cliff with elevations α and β respectively strike an object simultaneously at the same point. Show that if a is the horizontal distance of the object from the cliff, the height of the cliff is

$$a(\tan\beta-\tan\alpha). \qquad (P.U.\ 1961\ S)$$

12. Three particles are projected simultaneously in the same vertical plane from the same point ; one vertically, another at an elevation of 60° and the third horizontally. If their velocities are in the ratio $\sqrt{3} : 1 : 1$; show that they always lie in a straight line.

13. A triangle ABC, right-angled at C is placed in a vertical plane with AB horizontal and C uppermost. A particle fired from A passes exactly through C and B. Show that the angle θ and the velocity of projection u are given by

$$\tan\theta=\frac{c}{p} \text{ and } u^2=\frac{g(c^2+p^2)}{2p},$$

where $AB=c$ and p is the altitude of the triangle.

14. A stone is projected with velocity V and elevation θ from a point O on level ground so as to hit a mark P on a wall whose distance from O is a, the height of P above the ground being b. Prove that

$$2V^2(a\sin\theta\cos\theta-b\cos^2\theta)=ga^2.$$

Also prove that the requisite velocity of projection is least when

$$\theta=\frac{\pi}{4}+\frac{\alpha}{2}\cdot$$

where α is the elevation of P as seen from O.

15. A particle is projected so as just to clear a wall of height b at a horizontal distance a, and to have a range c from the point of projection, show that velocity of projection V is given by

$$\frac{2V^2}{g}=\frac{a^2\,(c-a)^2+b^2c^2}{ab(c-a)}\cdot$$

16. A gun is firing from the sea level out to sea. It is then mounted in a battery h feet higher up and fired at the same elevation α. Show that the range is increased by

$$\frac{1}{2}\left[\left(1+\frac{2gh}{v^2\sin^2\alpha}\right)^{\frac{1}{2}}-1\right]$$

of itself, v being the velocity of projection. *(P.U.)*

17. A stone is thrown in such a manner that it would just hit a bird at the top of a tree and afterwards reach a height double that of the tree; if, at the moment of throwing the stone, the bird fly away horizontally, show that, notwithstanding this, the stone will hit the bird if its horizontal velocity be to that of the bird as $(\sqrt{2}+1):2$.

(Agra, 1950)

18 A shell fired with velocity V at elevation θ, hits an airship at a height H which is moving horizontally away from the gun with velocity v, show that if

$$(2V\cos\theta-v)(V^2\sin^2\theta-2gH)^{\frac{1}{2}}=vV\sin\theta,$$

the shell must have hit the airship if the latter had remained stationary in the position it occupied when gun was actually fired.

(Panjab Hons.)

19. The difference between the radii of the front and hind wheels of a carriage is d, and c is the distance between the axle-trees; a particle of mud, driven from the highest point of the hind wheel falls on the highest point of the front wheel. Show that the velocity of the carriage is

$$\sqrt{\frac{(c+d)(c-d)g}{4d}}.$$

[*See Solved Ex. 13*]

20. Show that in general there are two directions in which a particle may be projected with a given velocity from a given point A so as to pass through another given point B.

If t and t_1 be the times of flight from A to B and α the inclination of AB to the horizontal, prove that $t^2+2tt_1\sin\alpha+t_1^2$ is independent of α. *(Agra, 1955 ; Banaras, 55 ; Panjab Hons.)*

ANSWERS

Exercise V (A), Page 101

1. 80 f.p.s. ; $\frac{5\sqrt{2}}{2}$.

2. $a = \tan^{-1} \frac{4}{3}$; $u = 200$ f.p.s. ; $T = 10$ sec.

4. $u = 80\sqrt{110}$ f.p.s. ;

Vel. at the highest point $= u \cos \alpha = 48\sqrt{110}$ f.p.s.

5. 792 ft. **7.** $200\sqrt{3}$ feet.

12. (*i*) Let $u \cos \alpha = K$, then the locus of the foci is

$$x^2 = \frac{2K^2}{g}\left(y + \frac{K^2}{2g}\right), \text{ and etc.}$$

Exercise V (B), Page 106

1. (*a*) $v = 16\sqrt{13}$ f.p.s. (*b*) $V = 32\sqrt{3}$ f.p.s. (*c*) $V' = 56$ f.p.s.

(*a*) $\theta = \tan^{-1} \frac{1}{2\sqrt{3}}$ for v. (*c*) $\theta = \tan^{-1} \frac{1}{4\sqrt{3}}$ for V' ; $\frac{1}{2}$ sec.

2. $T = (3\sqrt{2} + 2)$ sec. ; $R = (278 + 96\sqrt{2})$ feet.

3. $u = 64\sqrt{3}$ f.p.s., $\alpha = 60°$.

Exercise V (C), Page 112

1. $u = 177\frac{7}{9}$ f.p.s. **2.** $\pi/4$ and $\tan^{-1} 3$.

3. $\tan \theta = \frac{1058}{1425}$; $v = \frac{2}{15} \frac{\sqrt{3149989 \times 2}}{19}$

4. $u = 32\sqrt{13}$ f.p.s. ; $\alpha = \tan^{-1} \frac{2}{3}$.

5. $h = \frac{xy}{x + y}$. **7.** $u = 60$ f.p.s.

8. $u = 10\sqrt{10}$; $\alpha = \tan^{-1} 3$.

Exercise V (D), Page 118

1. 2000 m. and 6000 m.

2. 16875 feet ; $T = 32{\cdot}5$ sec. approx.

8. $\tan^{-1} \frac{1}{2}$.

Miscellaneous Exercise V, Page 120

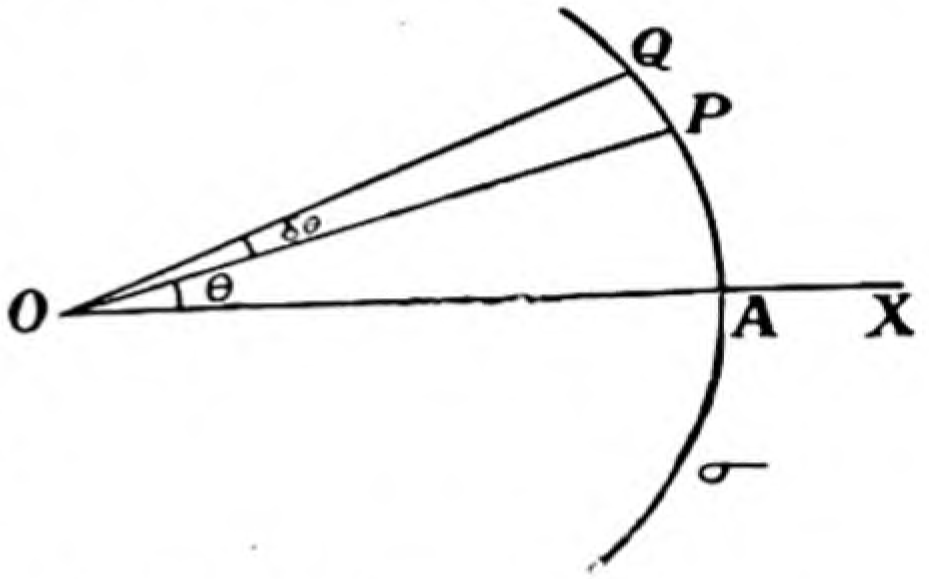
5. $2\left(h + \frac{r^2}{16h}\right)$.

CHAPTER VI

MOTION IN A HORIZONTAL CIRCLE

6·1 Angular Velocity and Angular Acceleration :

Let OX be a fixed axis in a plane with the fixed origin O. Suppose a particle P moves in a plane curve σ, and the line joining P to O makes angle θ with OX, at any instant t. Then rate of change of angle θ *w.r.t.* time is defined the "*angular velocity*" of the particle P, about O (or of the line OP) while the rate of change of the angular velocity *w.r.t.* time is known as the "*angular acceleration.*"



The Angular Velocity is generally denoted by ω.

6·11 Expressions for the Angular Velocity and Angular Acceleration.

Let a particle P move in a plane curve σ so that OP, the line joining it to the fixed origin rotates through an angle θ (measured in radians) in time t.

Let it further rotate through an infinitesimally small angle $\delta\theta$ in an interval δt ; then the average angular velocity

$$=\frac{\delta\theta}{\delta t}.$$

As $\delta t \to 0$, $\delta\theta$ also $\to 0$, and the limiting value of

$$\frac{\delta\theta}{\delta t} \quad i.e., \ \operatorname*{Lt}_{\delta \to 0} \frac{\delta\theta}{\delta t} \text{ or } \frac{d\theta}{dt}$$

is defined as the angular velocity of P about O.

Thus
$$\omega=\frac{d\theta}{dt}.$$

Further, angular acceleration

$$=\frac{d}{dt}(\omega)=\frac{d^2\theta}{dt^2}.$$

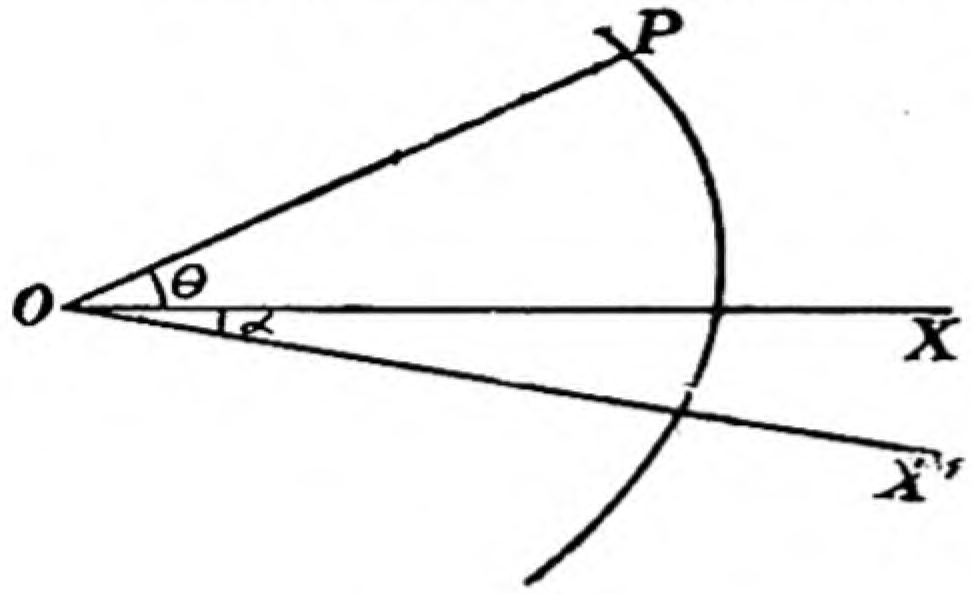
Also, $\because$
$$\frac{d\omega}{dt}=\frac{d\omega}{d\theta}\times\frac{d\theta}{dt}=\omega\times\frac{d\omega}{d\theta}.$$

Hence, angular acceleration may be denoted by

$$\frac{d\omega}{dt} \text{ or } \frac{d^2\theta}{dt^2} \text{ or } \omega\,\frac{d\omega}{d\theta}.$$

6·12 *Angular Velocity and Angular Acceleration are independent of the choice of the* **Initial line.**

For if OX′ is another line making angle α with OX ; then the angular velocity ω′ of P, about O, *w.r.t.* initial line OX′ is



$$\frac{d}{dt}(\angle X'OP)=\frac{d}{dt}(\theta+\alpha)$$
$$=\frac{d\theta}{dt}$$
$$=\omega.$$

Similarly for the Angular Acceleration.

6·13 Units for Angular Velocity and Acceleration.

Since θ is measured in radians, ω or $\frac{d\theta}{dt}$ is measured in radians per second, while $\frac{d^2\theta}{dt^2}$ is measured in radians per second per second.

Thus we may say that a flywheel is rotating with an angular velocity of 4 radians per second, so that

$$\omega=4 \text{ radians/second or } 4 \text{ rad./sec.}$$

If another wheel was rotating with 4 revolutions per second, then its angular velocity would be 8π rad./sec.

A velocity of r revolutions/min. $=\frac{2\pi r}{60}$ rad./sec.

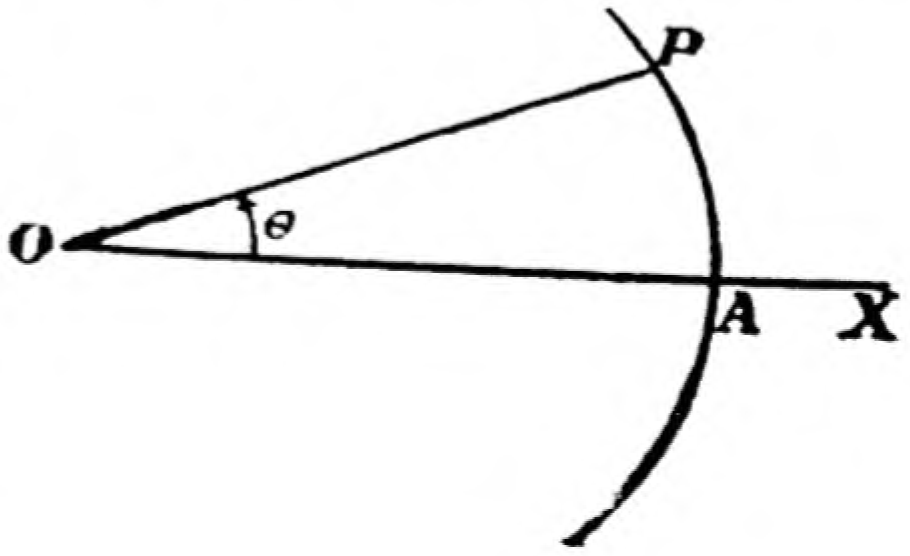
6·14 Motion with a constant Angular Acceleration.

Suppose a particle or a wheel is rotating about a fixed centre, with a constant angular acceleration α (a rad./sec^2.)

Let us take the line joining the origin O to its initial position A (at the instant of the first observantion) as the initial line.

Let the angular velocity at A be ...(I)

The problem is to find the position and the angular velocity after a certain time t.



By definition :

$$\frac{d^2\theta}{dt^2} \text{ or } \frac{d\omega}{dt}=a$$

Integrating we get

$$\omega=\alpha.t+c$$

Since $\omega=\omega_0$ when $t=0$, $\therefore\ c=\omega_0$

whence $\omega=\omega_0+\alpha.t$ ...(II)

or $\dfrac{d\theta}{dt}=\omega_0+\alpha.t.$

Integrating further, we get

$$\theta=\omega_0 t+\tfrac{1}{2}\alpha.t^2+c'$$

But when $t=0$, particle was at A, *i.e.* $\theta=0$.

Hence $\theta=\omega_0 t+\frac{1}{2}\alpha.t^2$ ...(III)

Eliminating t from (II) and (III), we obtain

$$\theta=(\omega_0+\tfrac{1}{2}\alpha t)\ t=[\omega_0+\tfrac{1}{2}(\omega-\omega_0)]\ \frac{\omega-\omega_0}{\alpha}$$

or $\omega^2=\omega_0{}^2+2\alpha\,\theta$ ...(IV)

Equation	*Units*
(II) gives ω in terms of t	radians/second
(III) ,, θ ,, ,, ,, t	radians
(IV) ,, ω ,, ,, ,, θ	radians/second

Ex. 1. *A flywheel is making 240 revolutions per minute, and after half a minute it is making 150 rev./min. How many revolutions will it make before coming to rest if the angular retardation is kept uniform? What time would have elapsed before it comes to rest?*

(*i*) Here $\omega_0=\dfrac{240\times 2\pi}{60}=8$ rad./sec.

$$t=30 \text{ sec.},\quad \omega=\frac{150\times 2\pi}{60}=5\pi \text{ rad./sec.}$$

If α is the uniform angular retardation, then from

$$\omega=\omega_0+\alpha.t$$

we have, $5\pi=8\pi+30.\alpha$ or $\alpha=-\dfrac{\pi}{10}$ rad./sec^2.

whence from $\omega^2=\omega_0{}^2+2\alpha\theta$, we get

$$0^2=(8\pi)^2-2\ \left(\frac{\pi}{10}\right)\theta$$

or $\theta=$**320π radians** or **160 revolutions.**

(*ii*) Also, t can be found from (II) or (III),

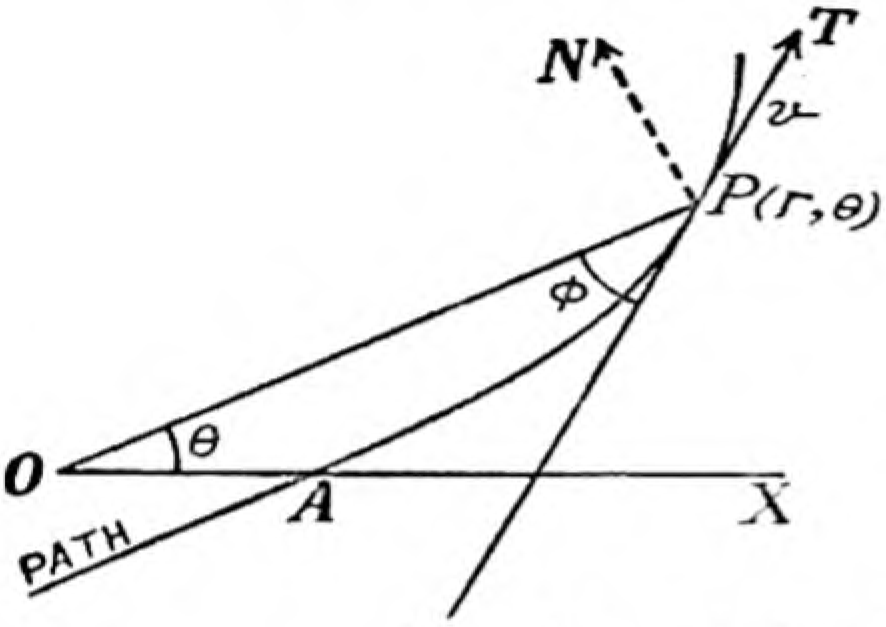
$$\omega=\omega_0+\alpha.t$$

$$0=8\pi-\frac{\pi}{10}t$$

$\therefore$ $\qquad t=$**80 seconds.**

6·2 Relation between the Angular Velocity and Linear Velocity. (*P.U.*)

Let OX be the initial line and let v and ω be the linear velocity (which is always along the tangent) and the angular velocity respectively at any instant t.



Let the tangent at P to the path of the particle, make an angle ϕ with the radius vector OP.

Then, ω, the angular velocity of P about O

$$=\frac{d\theta}{dt}=\frac{d\theta}{ds}\cdot\frac{ds}{dt}$$

$$=v\cdot\frac{1}{r}\cdot r\frac{d\theta}{ds}=\frac{v\sin\phi}{\gamma}$$

$$\left[\because \quad r\frac{d\theta}{ds}=\sin\phi \text{ (Differential Calculus)}\right]$$

$$=\frac{\text{Resolved part of } v \perp \text{OP}}{\text{OP}} \qquad (\because \quad \angle\text{TPN}=90°-\phi)$$

Cor. 1. If the path is a circle and O is at the centre

$$\text{OP}\perp\text{PT}$$

so that $\qquad \phi=\pi/2$ always.

Then, ω, the angular velocity of a point moving in a circle about the centre$=\dfrac{\text{Res. part of } v \perp \text{OP}}{\text{OP}}$

$$=\frac{v\sin\pi/2}{r}$$

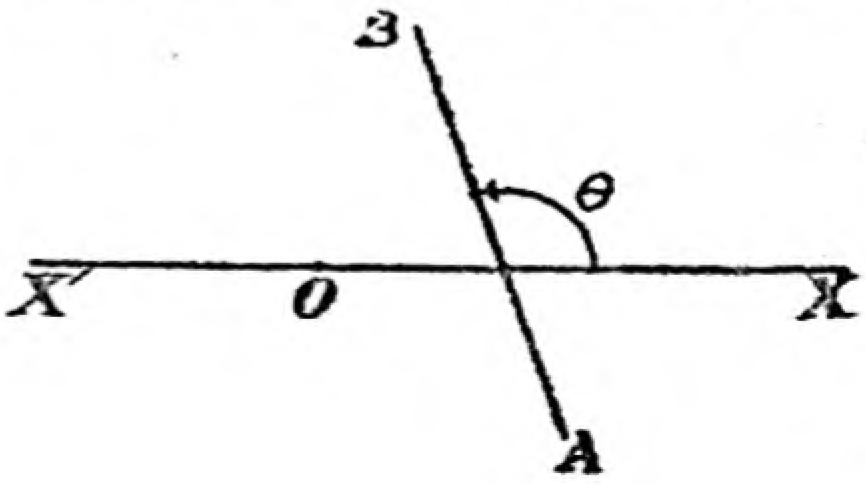
Thus $\quad \omega=\dfrac{v}{r}$

where r is the radius of the circle.

6·21 Angular Velocity of a line moving in a plane.

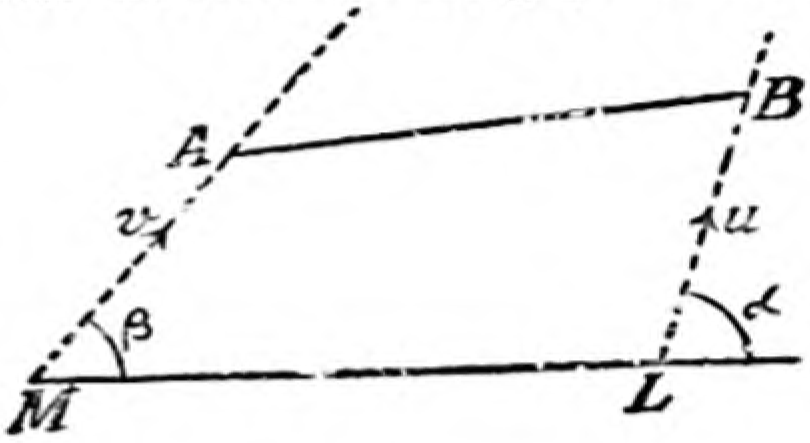
Let a moving line AB make an angle θ with any fixed line XOX′ in the plane at any instant then $\dfrac{d\theta}{dt}$ is defined as *the angular velocity of the line* AB.

Note : The angular velocity (of rotation) of AB in the plane is independent of the choice of *fixed axis* X′OX.



6·22 Angular Velocity of the line joining two points moving in a plane.

Let L and M be the initial positions of two moving particles A and B and let their velocities u and v make angles α and β with LM.



Then, ω, the angular velocity of the line joining A, B or the angular velocity of B relative to A

$$=\frac{\text{comp. of vel. of B relative to A} \perp \text{to AB}}{\text{AB}}$$

$$=\frac{\text{difference of vel. components} \perp \text{LM}}{\text{AB}}$$

$$=\frac{u \sin \alpha - v \sin \beta}{\text{AB}}.$$

6·23 Cor. If AB remains constant (equal to a), then A must have the same velocity as B in the direction of AB, *i.e.*

$$u \cos \alpha = v \cos \beta$$

or

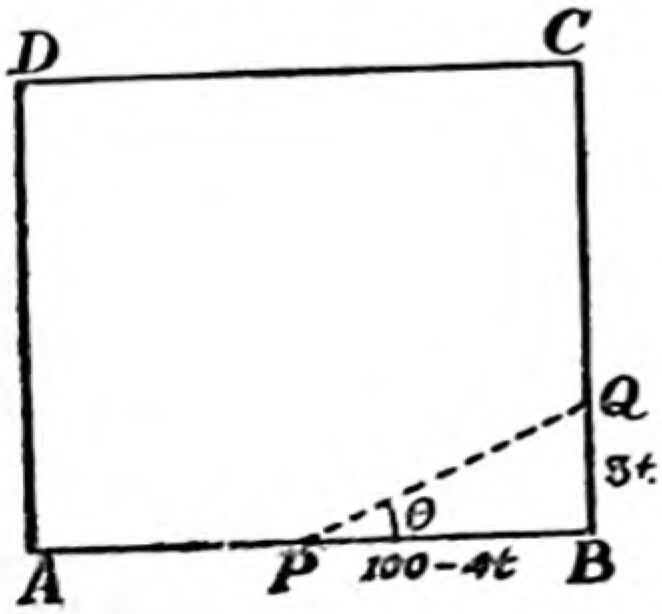
$$v=\frac{u \cos \alpha}{\cos \beta}$$

whence, the Angular Velocity of AB

$$=\frac{u \sin \alpha - \dfrac{u \cos \alpha}{\cos \beta} \cdot \sin \beta}{\text{AB}}$$

$$=\frac{u \sin (\alpha-\beta)}{a \cos \beta}.$$

Ex. 2. *ABCD is a square of side 100 ft. A person P starts from A and walks towards B with a speed of 4 f.p.s. At the same instant Q starts from B and walks towards C with a speed of 3 f.p.s. Show that the angular velocity of the straight line PQ, after t seconds from the instant of starting is* $\dfrac{12}{t^2-32t+400}$ *radians/sec., and that it is the greatest when PQ is 60 ft.* (*P.U.*)



After t seconds,

$\text{AP}=4t$ ...(I)

$\text{BQ}=3t$ ...(II)

$\therefore \quad \text{PB}=100-4t$ ...(III)

Let PQ make angle θ with AB

$$\tan\theta=\frac{3t}{100-4t}$$

Differentiating *w.r.t. t*,

$$\sec^2\theta\,.\,\frac{d\theta}{dt}=\frac{(100-4t)\,.\,3-(3t)(-4)}{(100-4t)^2}=\frac{300}{(100-4t)^2}$$

whence
$$\omega=\frac{d\theta}{dt}\quad\frac{300}{(100-4t)^2}\cos^2\theta$$

$$=\frac{300}{(100-4t)^2}\times\frac{(100-4t)^2}{(3t)^2+(100-4t)^2}$$

$$=\frac{\mathbf{12}}{\mathbf{t^2-32t+400}}$$

(*ii*) Further, ω is maximum, when

$$t^2-32t+400=\text{minimum}$$

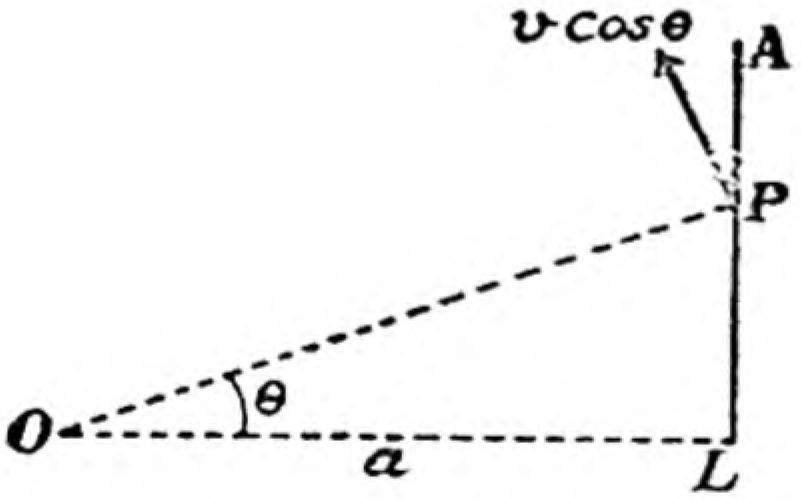
i.e.
$$\frac{d}{dt}(t^2-32t+400)=0 \text{ and } \frac{d^2}{dt^2}(t^2-32t+400)>0$$

i.e.
$$2t=32 \quad \text{or} \quad t=16$$

For $t=16$, $PB=36$, $BQ=48$

$\therefore$
$$PQ=\sqrt{PB^2+BQ^2}=\mathbf{60\ ft.}$$

Ex. 3. *A point moves uniformly along a given straight line; show that its angular velocity about a fixed point varies inversely as the square of its distance from the fixed point.*



Let AB be the fixed line along which the particle moves. Let O be the fixed point and P be the position of the particle on the line AB at any time *t*. Let OL($=a$) be the perpendicular from O to the line AB. The point O and the line AB being fixed, *a* is constant.

Let the linear velocity of P be *v*.

Then angular velocity of P about O

$$=\frac{\text{Velocity perpendicular to OP}}{OP}$$

$$=\frac{v\cos\theta}{OP}=\frac{v}{OP}\cdot\frac{a}{OP}=\frac{av}{OP^2}$$

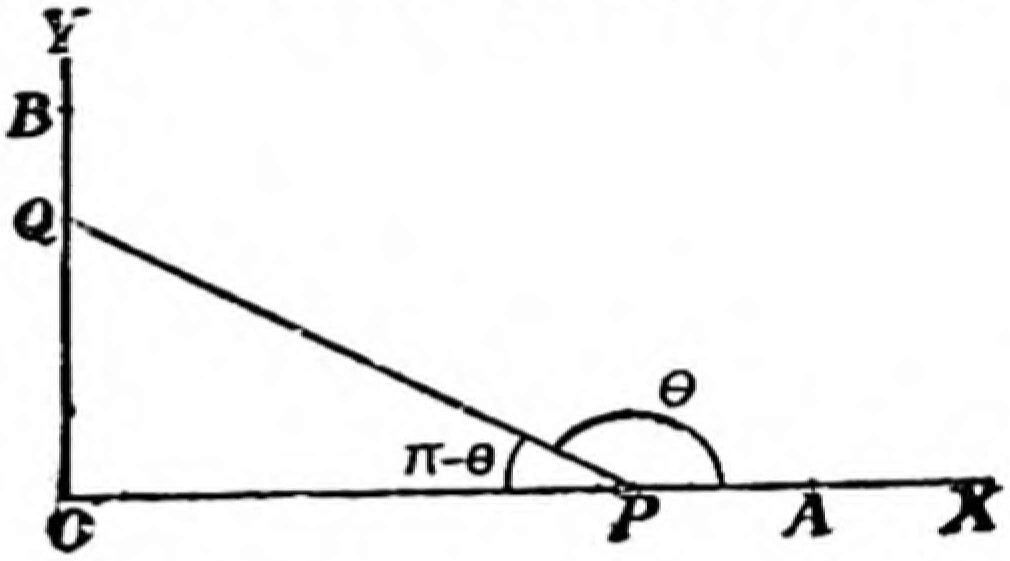
which varies as $\frac{1}{OP^2}$.

Ex. 4. *Two points are moving with uniform velocities u and v in perpendicular lines OX and OY, the motions being towards O.*

If initially, their distances from the origin are a and b respectively, calculate the angular velocity of the line joining them at the end of t secs. and show that it is greatest when

$$t=\frac{au+bv}{u^2+v^2}\cdot \qquad (P.U.)$$

Let A and B be the initial positions of the particles where OA$=a$ and OB$=b$.



Let P and Q be the positions of the particles after time t. Since the particles are moving with uniform velocities u and v, we have

$$\text{AP}=ut \text{ and } \text{BQ}=vt.$$

$$\therefore \quad \text{OP}=a-ut$$

and $$\text{OQ}=b-vt.$$

Let θ be the angle that PQ makes with OX.

Then $$\tan(\pi-\theta)=\frac{\text{OQ}}{\text{OP}}=\frac{b-vt}{a-ut}$$

or $$\tan\theta=-\frac{b-vt}{a-ut}$$

$$\therefore \qquad \theta=\tan^{-1}\frac{vt-b}{a-ut}$$

$$\omega=\frac{d\theta}{dt}=\frac{1}{1+\left(\dfrac{vt-b}{a-ut}\right)^2}\cdot\frac{(a-ut)v-(vt-b)(-u)}{(a-ut)^2}$$

$$=\frac{av-bu}{(a-ut)^2+(vt-b)^2}$$

ω is greatest when $(a-ut)^2+(vt-b)^2$ is least.

Let $$y=(a-ut)^2+(vt-b)^2$$

$$\frac{dy}{dt}=2(a-ut)(-u)+2(vt-b)\cdot v$$

y is least if $$\frac{dy}{dt}=0 \text{ and } \frac{d^2y}{dt^2} \text{ is positive}$$

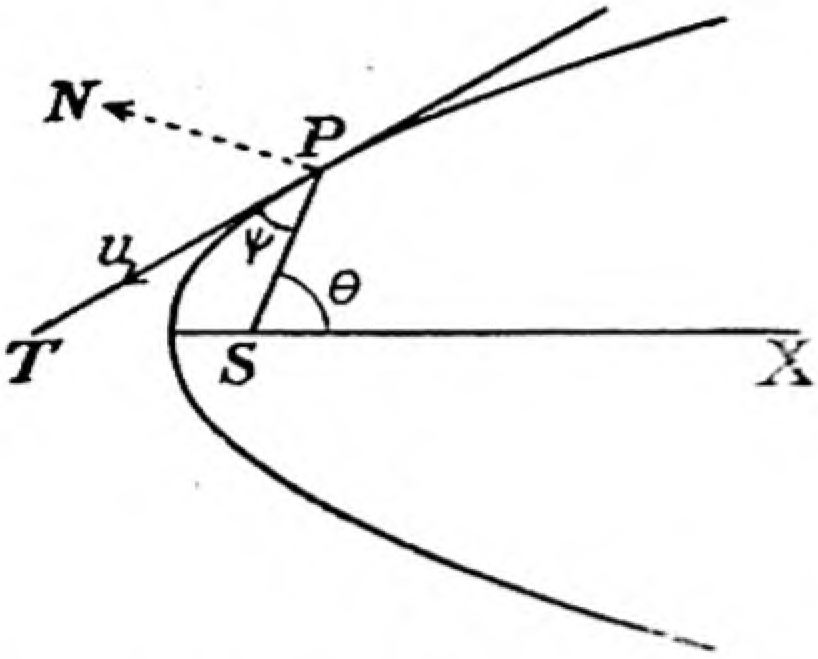
$$\frac{dy}{dt}=0 \text{ gives } t=\frac{au+bv}{u^2+v^2}$$

and $$\frac{d^2y}{dt^2}=2(u^2+v^2)>0.$$

$\therefore$ y is minimum and hence $\dfrac{d\theta}{dt}$ is greatest

when $$t=\frac{(au+bv)}{(u^2+v^2)}\cdot$$

Ex. 5. *A point P is moving in a parabola with contant speed. Show that the angular velocity about the focus S varies as* $(SP)^{\frac{3}{2}}$.



Let a particle move with constant velocity v in a parabola. The direction of v at any instant is along PT, the tangent at P. Let S be the focus and SX the axis. Join SP and draw PN ⊥ SP.

Then ω, the angular velocity about the focus (S)

$$=\frac{\text{Resolved part of } v \text{ along PN}}{\text{SP}}$$

$$=\frac{v\sin\phi}{\text{SP}}. \qquad \ldots\text{(I)}$$

The polar equation of the parabola, *w.r.t.* S as the pole and SX as initial line is

$$\frac{l}{r}=1-\cos\theta$$

or

$$r=\frac{l}{1-\cos\theta}=\frac{l}{2}\operatorname{cosec}^2\frac{\theta}{2} \qquad \ldots\text{(II)}$$

On differentiation,

$$\frac{dr}{d\theta}=-\frac{l}{2}\operatorname{cosec}^2\frac{\theta}{2}\,.\,\cot\frac{\theta}{2}. \qquad \ldots\text{(III)}$$

By division, from II and III,

$$r\frac{d\theta}{dr}=-\tan\frac{\theta}{2}$$

or

$$\tan\phi=-\tan\frac{\theta}{2}=\tan\left(\pi-\frac{\theta}{2}\right)$$

whence

$$\phi=\pi-\frac{\theta}{2}$$

$$\therefore \quad \omega=\frac{v\sin\left(\pi-\frac{\theta}{2}\right)}{r}=\frac{v\sin\frac{\theta}{2}}{r}=\frac{v\sqrt{\frac{l}{2r}}}{r}$$

$$=\sqrt{\frac{v^2l}{2}}\Big/r^{\frac{3}{2}}$$

$$\therefore \quad \omega\propto\frac{1}{r^{\frac{3}{2}}}.$$

Exercise VI (A)

1. Define angular velocity and acceleration.

A point is moving in a circle. Show that at any instant its angular velocity about a point on the circumference is half its angular velocity about the centre. *(P.U.)*

2. A flywheel is making 150 revolutions per minute, and after 30 secs it is turning at 60 revolutions per minute. How many revolutions will it make and what time will elapse before stopping if the retardation is uniform ?

3. A wheel is moving with a uniform angular acceleration of π radians per second. How many revolutions per second will it be making at the end of one minute ?

If the effort is taken off and if the frictional resistance of the bearings is equivalent to a uniform retardation of $\frac{\pi}{4}$ radian per sec.^{2}, when will the wheel come to rest ?

4. A wheel rolls uniformly on the ground, without sliding, its centre describing a straight line ; find the velocity of any point of its rim and show that its angular velocity about the point of contact of the wheel with the ground is equal to the angular velocity of the wheel about it centre. [See Note 5·8]

5. A rod moves in a plane, so that its ends move in two perpendicular grooves OA, OB. Show that the projection of any fixed point on OA (or OB) executes simple harmonic motion.

6. Prove that the angular velocity of a projectile P about the focus S of its path varies inversely as its distance from the focus S.

7. A particle falls freely under gravity from a point A, at a horizontal distance of a from O. Find the angular velocity of the line joining it to O, at any instant. Show that it is maximum after $\sqrt{\frac{2a}{g\sqrt{3}}}$ seconds and the particle has fallen a vertical distance $\frac{a}{\sqrt{3}}$.

8. A circular disc of radius a is free to turn round its centre in its own plane and a thin string is wound round its rim. If the free end A of the string is withdrawn along a radius with speed v, find the angular velocity of the disc when the distance of A from the centre is x.

9. A particle moves along the perimeter of a regular n-gon with uniform speed r. If r be the circumradius of the polygon, prove that at each corner of its angular points, the particle receives an additional velocity towards the circumcircle, of magnitude $\frac{av}{r}$, where a is the side of the polygon. Deduce by making $n \to \infty$, that the acceleration of a particle moving uniformly in a circle is v^2/r. *(P.U., 1961)*

10. (*a*) If two points describe the circle of radius r in the same sense and with the same speed v, show that the relative angular velocity of each *w.r.t.* the other is v/r.

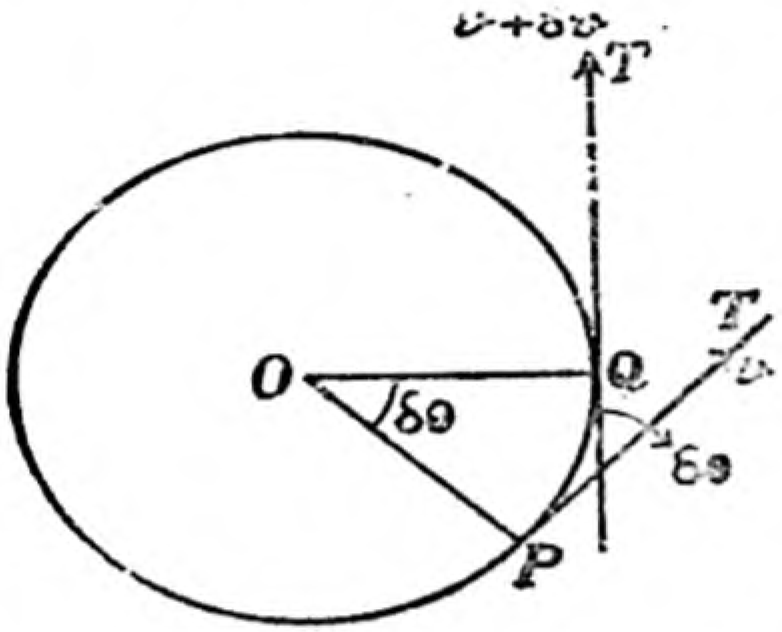
(*b*) Two particles are describing concentric circles of radii a and b with angular velocities ω, ω' respectively. Prove that the angular velocity of the line joining them when it is of length r is

$$\frac{(r^2+a^2-b^2)\omega+(r^2-a^2+b^2)\omega'}{2r^2}$$

(*P.U., 1961 ; Roorkee 1957*).

6·3 Expressions for Tangential and Normal Acceleration for Motion along a circle.

Let a particle be moving in a circle of radius r and centre O. Let P be its position at time t and Q its position at time $t+\delta t$. Let $\angle POQ=\delta\theta$ and arc $PQ=\delta s$.



Now velocity at time t along the tangent at $P=v$ ...(I)

and velocity at time $t+\delta t$ along the tangent at $Q=v+\delta v$.

Its resolved part along the tangent at $P=(v+\delta v)\cos\delta\theta$ ...(II)

$\therefore$ change in velocity in time δt along the tangent at P

$$=(v+\delta v)\cos\delta\theta-v \qquad \text{...(III)}$$

$\therefore$ average rate of change of velocity along tangent at P

$$=\frac{(v+\delta v)\cos\delta\theta-v}{\delta t}$$

$\therefore$ tangential acceleration at P

$$=\underset{\delta t\to 0}{\text{Lt}}\ \frac{(v+\delta v)\cos\delta\theta-v}{\delta t}$$

$$=\underset{\delta t\to 0}{\text{Lt}}\ \frac{(v+\delta v)\left[1-\frac{(\delta\theta)^2}{2}+\ldots\ldots\right]-v}{\delta t}$$

(neglecting infinitesimals of higher orders)

$$=\underset{\delta t\to 0}{\text{Lt}}\ \frac{\delta v}{\delta t}=\frac{dv}{dt}=\frac{d}{dt}\left(\frac{ds}{dt}\right)=\frac{d^2s}{dt^2}$$

Hence Tangential Acceleration $=\dfrac{\mathbf{dv}}{\mathbf{dt}}$ or $\dfrac{\mathbf{d^2s}}{\mathbf{dt^2}}$ or $\mathbf{v}\dfrac{\mathbf{dv}}{\mathbf{ds}}$.

(*ii*) Again, the velocity at time along the normal $PO=0$.

And, the resolved part of the velocity at time $t+\delta t$ along the normal $PO=(v+\delta v)\sin\delta\theta$.

$\therefore$ change in the velocity in time δt along the normal at P

$$=(v+\delta v)\sin\delta\theta-0$$

$\therefore$ average rate of change of velocity along the normal at P

$$=\frac{(v+\delta v)\sin\delta\theta}{\delta t}$$

$\therefore$ normal acceleration at P

$$=\underset{\delta t\to 0}{\text{Lt}}\ \frac{(v+\delta v)\sin\delta\theta}{\delta t}$$

$$=\underset{\delta t\to 0}{\text{Lt}}\ \frac{(v+\delta v)\left[\delta\theta-\frac{(\delta\theta)^3}{6}+\ldots\ldots\right]}{\delta t}$$

$$=v\,\frac{d\theta}{dt}$$

$$=v.\frac{v}{r}\qquad\left(\because\ \frac{d\theta}{dt}=\omega=\frac{v}{r}\right)$$

Hence Normal Acceleration $=\dfrac{\mathbf{v}^2}{\mathbf{r}}$ **or** $\omega^2\mathbf{r}$.

(*Delhi, 1957 ; P.U. 1967*)

6·31 Mechanical Significance of the Tagential and Normal Accelerations.

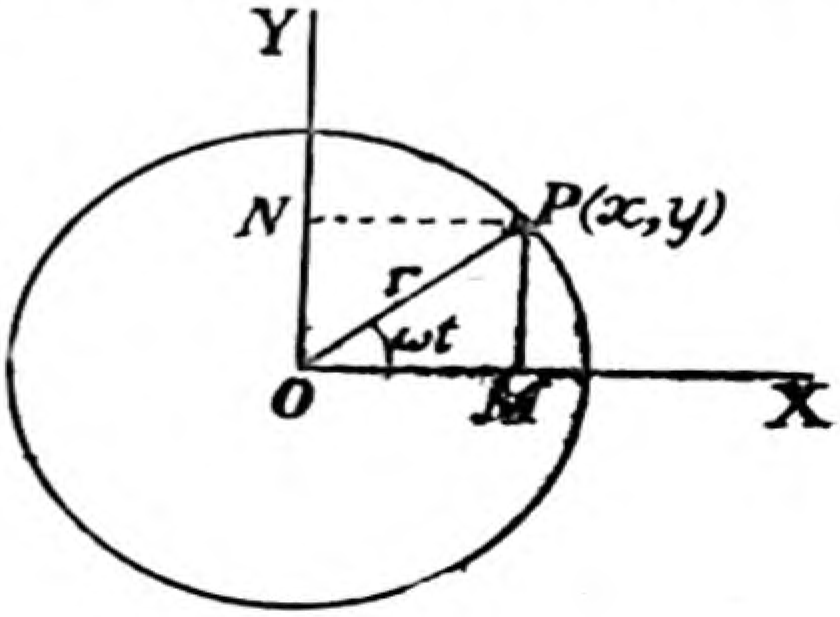
The *tangential component* of acceleration changes the velocity in magnitude while the *normal component* changes the direction of the velocity so that if the tangential acceleration is zero, the particle moves with a constant speed. But if the normal acceleration is zero, the motion will take place along a straight line, it cannot move in a circle.

The expression for the normal acceleration if the particle be moving along any curved path is $\dfrac{v^2}{\rho}$, where ρ is the radius of curvature of the path at P.

6·32 Another Method for obtaining the expression for acceleration when a particle moves in a circle with uniform speed.

(*Delhi 1957 ; P.U. B.Sc. 1962*)

At any time t, let P be the position of the particle moving uniformly in a circle of radius r and centre O. Let the co-ordinates of P be (x, y) referred to OX and OY as perpendicular axis.



Then since the particle moves along the circle with uniform speed, its angular velocity $\omega\left(=\dfrac{v}{r}\right)$ about the centre O will also be uniform.

Let the particle describe the angle AOP in time t.

Then $\angle AOP=\omega t.$

$$\therefore \quad x=r\cos\omega t \qquad\qquad y=r\sin\omega t$$

$$\therefore \quad \frac{dx}{dt}=-r\omega\sin\omega t \qquad\qquad \frac{dy}{dt}=r\omega\cos\omega t$$

$$\frac{d^2x}{dt^2}=-r\omega^2\cos\omega t \qquad\qquad \frac{d^2y}{dt^2}=-r\omega^2\cos\omega t$$

$$=-\omega^2x \quad \ldots\text{(I)} \qquad\qquad =-\omega^2y \quad \ldots\text{(II)}$$

$$\therefore \quad \text{Resultant acceleration at P}=\sqrt{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2}$$

$$=r\omega^2$$

$$=r.\,\frac{v^2}{r^2}$$

$$=\frac{v^2}{r} \qquad \ldots\text{(III)}$$

If θ be the angle that the resultant acceleration makes with OX, then

$$\tan\theta=\frac{d^2y}{dt^2}\bigg/\frac{d^2x}{dt^3}$$

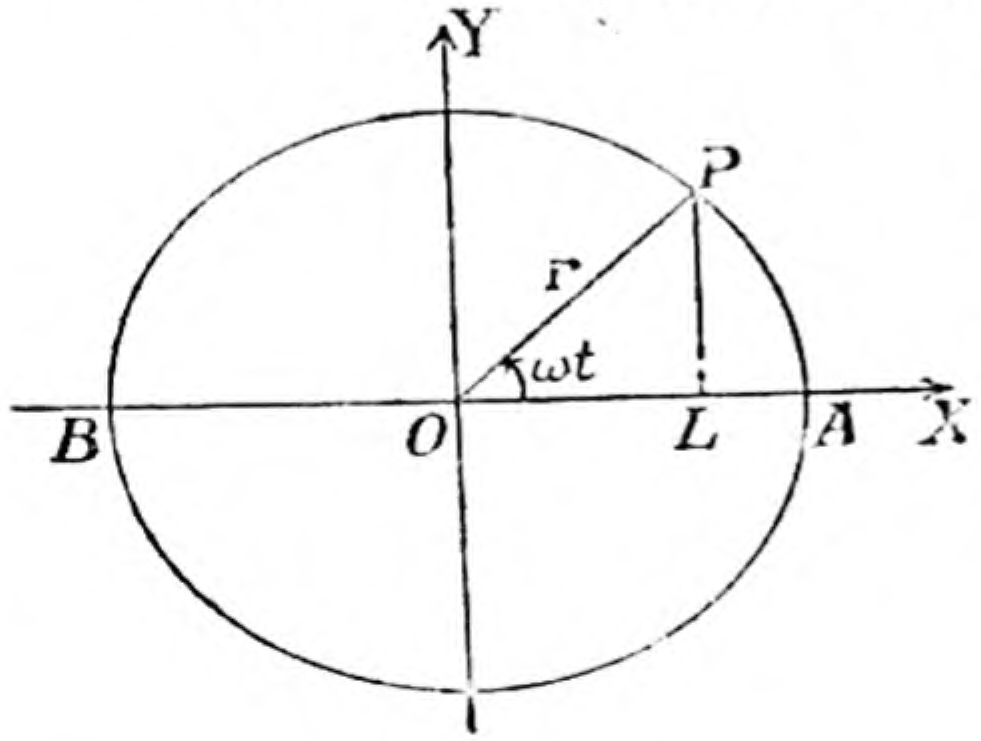
$$=\tan\omega t$$

$$\therefore \qquad \theta=wt=\angle AOP \qquad \ldots\text{(IV)}$$

and since both components of the acceleration are negative, the acceleration must be along PO.

6·33 *A particle describes a circle with constant angular velocity, to prove that the foot of the perpendicular from it on any diameter executes a simple harmonic motion.* *(P.U., 1954)*

Let P be the position of the particle at time t and L be the foot of the perpendicular from P on the diameter AB.



Let r be the radius of the circle. Since the particle is moving with a constant angular velocity ω (radians per second), the angle AOP described in time $t=\omega t$ radians.

$$\text{Let } x=OL=OP\cos\angle AOP$$

$$=r\cos\omega t$$

$$\therefore \quad \frac{dx}{dt}=-r\omega\sin\omega t$$

(ω being constant)

$$\text{and} \qquad \frac{d^2x}{dt^2}=-r\omega^2\cos\omega t$$

$$=-\omega^2x$$

which is the equation of simple harmonic motion.

6·34 Equations of Motion for a Circular Motion.

When a particle of mass m moves along a circle of radius r; then since the Tangential and Normal Accelerations are

$$v\frac{dv}{ds} \quad \text{and} \quad \frac{v^2}{r} \text{ (or } \omega^2 r\text{)};$$

the Tangential and Normal force are

$$mv\frac{dv}{ds} \quad \text{and} \quad \frac{mv^2}{r} \text{ (or } m\omega^2 r\text{)}.$$

In order that the particle must describe a circle of radius r, the force required to supply the normal acceleration

$$\frac{v^2}{r} \text{ is } \frac{mv^2}{r}$$

or $mr\omega^2$ (measured in absolute units). This force, which acts towards the centre of the circle and known as *centripetal force* may be produced in various ways.

If a particle is attached to one end of a string, the other end of which is held in hand, and the particle is swinging round, the necessary central force is supplied by the tension of the string. Although the string is in the state of tension, there is no tendency of the particle to move away along the radius of the circle. If the string breaks, the particle will continue to move along the tangent to the circle.

In case of a train moving round a curved path, the necessary inward force is provided by the reaction of the rails on the flanges of the wheels. In case of a motor car, this force is supplied by the force of friction between the wheels and the ground.

Ex. 6. *A particle of mass 8 lbs., resting on a smooth table and attached to a fixed point on the table by a string 4 ft. long, is making 300 revolutions per minute; find the tension in the string.*

Let

$$\begin{aligned}\omega &= 300 \text{ r.p.m.}\\ &= \frac{300}{60} \text{ r.p.s.}\\ &= 5\times 2\pi \text{ radians per sec.}\\ &= 10\pi \text{ radians/sec.}\end{aligned}$$

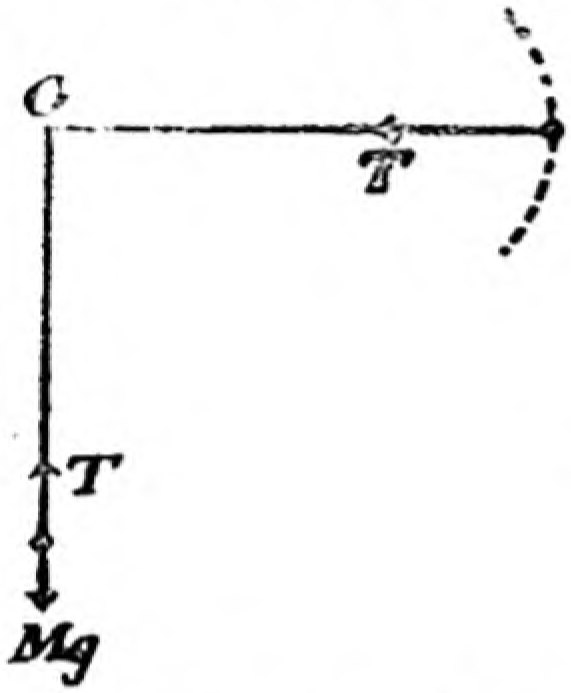
Since the necessary centripetal force $m\omega^2 r$ is supplied by the tension,

$\therefore$ tension in the string

$$\begin{aligned}&= mr\omega^2\\ &= 8.4.(10\pi)^2 \text{ poundals}\\ &= \frac{8.4.10^2\ \pi^2}{32} \text{ lbs. wt.}\\ &= 100\pi^2 \text{ lb. wt.}\\ &= 1000 \text{ lb. wt. nearly.}\end{aligned}$$

Ex. 7. *A particle of mass m is fastened to one end of a string passing through a hole in a smooth table and supports at its other end a particle of mass M. Find the number of revolutions per sec. which the particle on the table must make in a circle of radius r ft. in order to keep the other particle at rest.*

Let the angular velocity of the particle describing a circle of radius r on the table be ω.



Since the string passes through a smooth hole, the tension T of the string throughout its length is the same.

The particle of mass M will hang in rest

$$\text{if} \qquad T = Mg$$

$$\text{or} \qquad mr\omega^2 = Mg$$

$$\therefore \qquad \omega = \sqrt{\frac{Mg}{mr}}.$$

$\therefore$ number of revolutions per sec.

$$= \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Mg}{mr}}.$$

Exercise VI (B)

1. A particle is moving along a circle of radius a such that the actual distance of the particle at any time t from a fixed point of the circle is s. Show that the magnitude of the acceleration of the particle at time t is

$$\sqrt{\left(\frac{d^2s}{dt^2}\right)^2 + a^{-2}\left(\frac{ds}{dt}\right)^2}. \qquad (P.U.)$$

2. (*a*) A particle moves in a circle with uniform speed. Show that its projection on any diameter of the centre moves with Simple Harmonic Motion. (*P.U.*)

(*b*) A point P moves in a circle with uniform angular velocity ω about the centre C. If ϕ be the orthogonal projection of P upon a fixed diameter of the circle, determine the motion of ϕ and find its velocity. (*P.U.*)

3. A particle of mass 2 lbs. describes a circle in a horizontal plane, round a point with which it is connected by a string of length 4 ft. If, in one second, the particle makes seven revolutions, find the normal acceleration and the tension of the string. (*P.U.*)

4. Four masses, each of 10 lb., are fastened together with four strings each 3 ft. long so as to form a square. This square rotates in a horizontal plane on a smooth table at a speed of one revolution per second ; find in pounds weight the tension in the string.

5. Two particles each of mass m, are attached to the ends of a light string of length a and placed on a horizontal turn-table at distances a, $2a$ from the centre of rotation. Prove that when the table rotates with angular velocity ω the particles can remain upon it without slipping provided that the coefficient of friction is not less than $\frac{3a\omega^2}{2g}$.

6. A particle of mass m moves on a horizontal table and is connected by a string of length l with a fixed point on the table. If the greatest weight that the string can support be that of a mass of M lb.; find the greatest number of revolutions that the particle can make without breaking the string. [See Ex. 7]

7. Two particles of masses m and $3m$ are fastened respectively to the middle point and one extremity of a light string which is laid on a smooth horizontal table, the other extremity of the string being kept fixed. The masses are projected so that they revolve in circles with uniform speeds and such that the two portions of the string remain in the same straight line. Show that the tensions in the two parts of the string ere as 7 : 6.

8. A particle of mass m on a smooth table is fastened to one end of a fine string which passes through a small hole in the table and supports at its other end a particle of mass $2m$, the first particle being held at a distance c from the hole. Find the velocity with which m must be projected so that it may describe a circle of radius c. [See Ex. 7]

9. A particle of mass m on a smooth table is fastened to one end of a fine string which passes through a small hole in the table and supports at its other end a particle of mass $4m$ which hangs vertically. Find the number of revolutions per minute which the particle on the table must perform in a circle of radius 6 inches in order to keep the other particle at rest. [See Ex. 7]

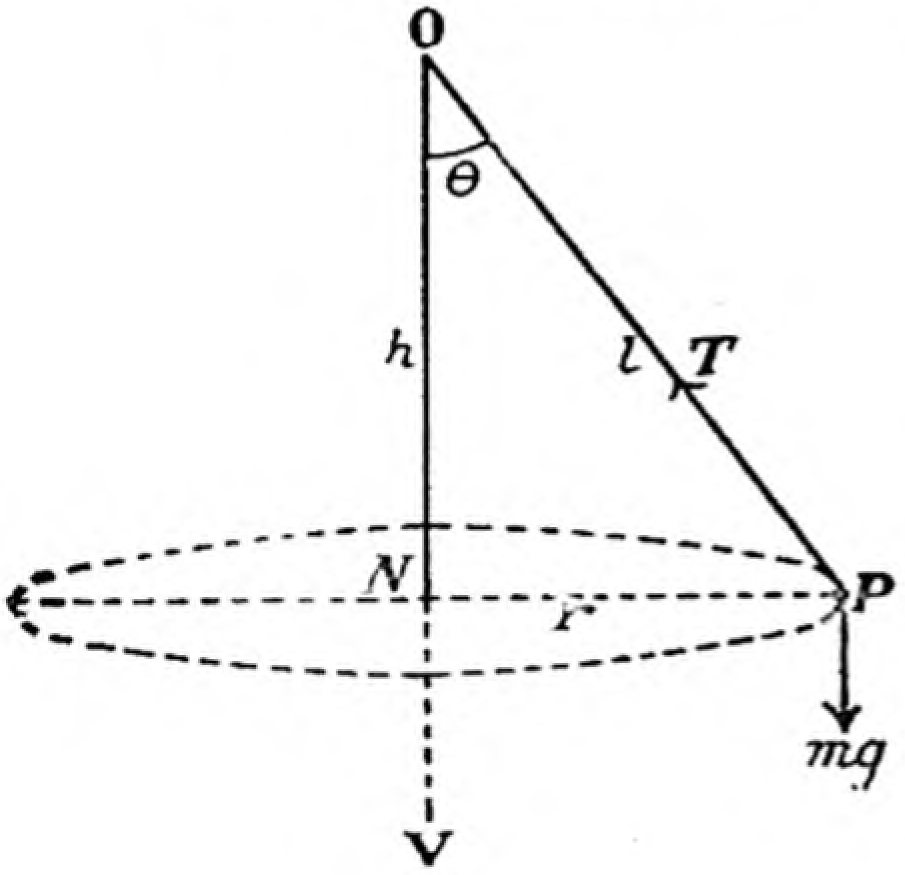
10. A body of mass m moves on a horizontal table being attatched to the fixed point on the table by an elastic string of modulus λ and natural length l. Find the velocity of the particle if it be describing a circle of radius r.

11. An elastic string of unstretched length l, fixed at one end, can just support a mass of m lb. when hanging vertically and extended by half its length. The mass and string are now placed on a smooth horizontal table with one end of the string fixed. The string is stretched to double its length and the mass is projected along the table with such velocity that it describes a horizontal circle about the fixed point as centre. Find the time of revolution of the mass.

12. A satellite (which may be regarded as a particle) of mass m pounds moves in a circle of radius 5,000 miles about the centre of the earth as centre (radius of earth$=$4,000 miles). The force in the particle is directed towards the earth's centre and varies inversely as the square of the distance from that point and is equal to $32m$ poundals at the surface of the earth. Find the velocity of the satellite and its time period. *(P.U.)*

6·4 Conical Pendulum. *If one end of a light string is fixed at a point O, and a particle tied to the other end moves in a horizontal circle, so that the string (OP) itself describes a right circular cone about the vertical through O (OV as the axis), the system is known as a* **Conical Pendulum.**

Let m be the mass of the particle tied at P, one end of the string whose other end O is fixed.



Let $OP=l$.

Draw $PN \perp OV$, the vertical through O.

Let $\angle NOP=\theta$.

As the string rotates, P moves in a horizontal circle with centre N and radius $NP=r=l\sin\theta$. ...(I)

The necessary normal force mw^2r to keep the mass in circular motion is supplied by the horizontal component of T, *i.e.*, $T\sin\theta$; while the vertical component $T\cos\theta$ balances mg, as there is no vertical motion.

$$\therefore \qquad mw^2r=T\sin\theta=T\,\frac{r}{l} \qquad \text{...(II)}$$

and $$mg=T\cos\theta \qquad \text{...(III)}$$

whence $$T=ml\omega^2.$$

$$\therefore \qquad \cos\theta=\frac{mg}{T}=\frac{g}{l\omega^2}. \qquad \text{...(IV)}$$

In order that the string may describe a cone,

$$0<\theta<\frac{\pi}{2} \quad \text{or} \quad 0<\cos\theta<1.$$

$$\therefore \qquad \omega^2>\frac{g}{l}$$

e., the least angular velocity required for the motion in a conical pendulum is $$\sqrt{\frac{g}{l}}. \qquad \text{...(V)}$$

Further $$h=ON=l\cos\theta=\frac{g}{\omega^2}.$$

If n is the number of revolutions per second in the motion of P, then

$$\omega = 2\pi n$$

so that

$$h = \frac{g}{4\pi^2 n^2} \qquad \text{...(VI)}$$

(*P.U. B.Sc., 1962*)

Again, from (IV),

$$\omega^2 = \frac{g}{l \cos\theta} = \frac{g}{h}. \qquad \text{...(VII)}$$

Time period for one revolution

$$= \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{h}{g}},$$

i.e., the time varies directly as the square root of the depth of the circle described below the point of suspension of the string. ...(VIII)

(*P.U., 1964*)

6·41 Constrained Motion in a Smooth Table.

Sometimes a particle moving in a conical pendulum is also constrained to move on a smooth horizontal plane (say table) ; then to the already present forces T and mg there is added another force R, the normal reaction of the smooth table on the mass.

The normal equation of motion as before is

$$m\omega^2 r = T \sin\theta = T \cdot \frac{r}{l} \qquad \text{...(I)}$$

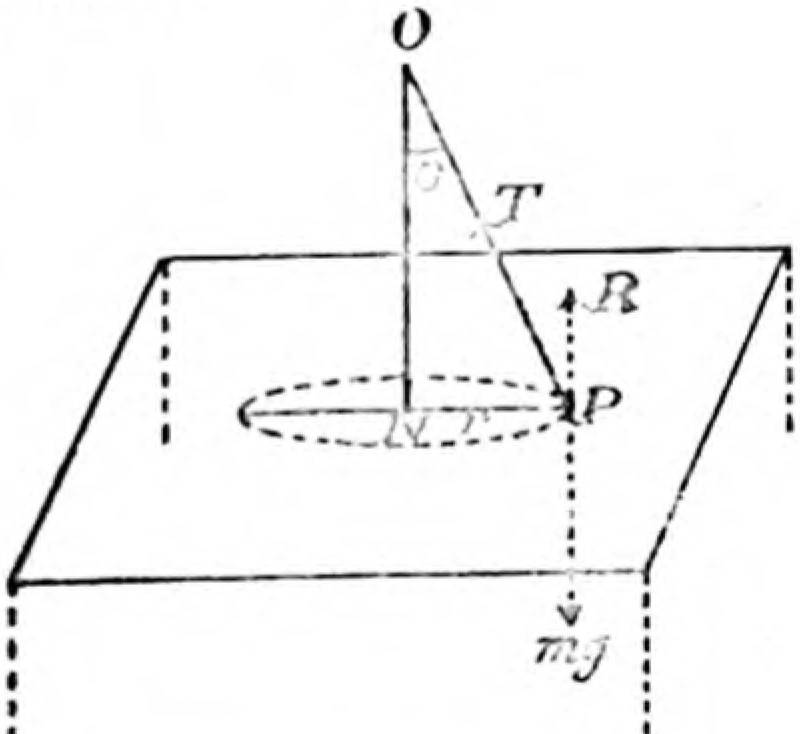
while resolving vertically we get

$$R + T \cos\theta = mg. \qquad \text{...(II)}$$

From (I),

$$\left.\begin{aligned} T &= ml\omega^2 \\ &= 4mln^2\pi^2 \end{aligned}\right\} \qquad \text{...(III)}$$

$$\begin{aligned} R &= mg - T \cdot \cos\theta \\ &= mg - 4mhn^2\pi^2. \end{aligned} \qquad \text{...(IV)}$$



In order that the particle may not leave the table, R must remain >0.

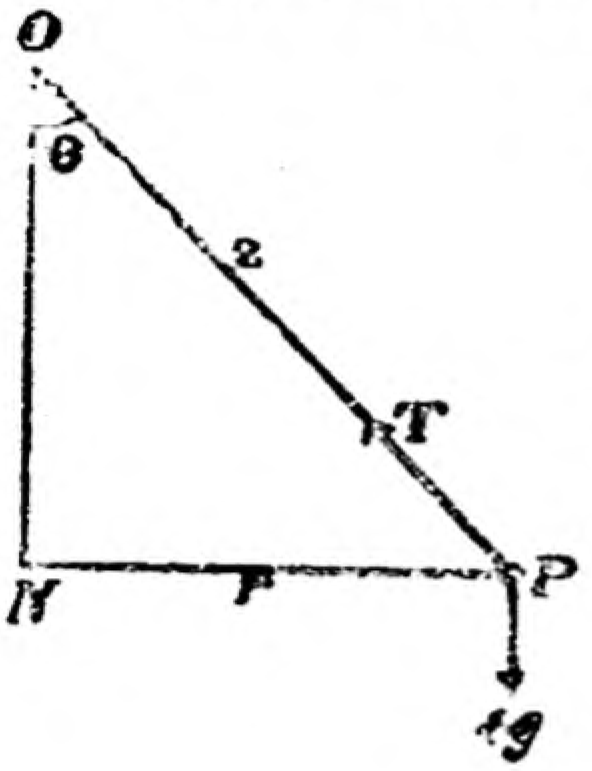
So that

$$4mh\pi^2 n^2 < mg$$

or

$$\omega^2 h < g \quad \text{or} \quad \omega < \sqrt{\frac{g}{h}}. \qquad \text{...(V)}$$

Ex. 8. *If the mass of the bob in a conical pendulum is 4 lb. and the length of the string is 2 ft., find the maximum number of revolutions per second of the pendulum when the greatest tension that can be allowed in the string is 40 lb. wt.*



Let θ be the inclination of the string to the vertical, T the tension in the string and ω the anuglar velocity. Let r be the radius of the horizontal circle described by the bob.

$$\text{Then } T \sin \theta = 4.r\omega^2$$
$$= 4 \times 2 \sin \theta . 4\pi^2 n^2$$

where n is the number of revolutions per second.

$$\therefore \quad T = 32\pi^2 n^2 \text{ poundals}$$
$$= \pi^2 n^2 \text{ lb. wt.}$$

Maximum value of T=40 lb. wt.

$$\therefore \quad \pi^2 n^2 = 40$$

$$\therefore \quad n^2 = \frac{40}{\pi^2} = 4 \text{ where } \pi^2 \text{ is taken} = 10 \text{ approx.}$$

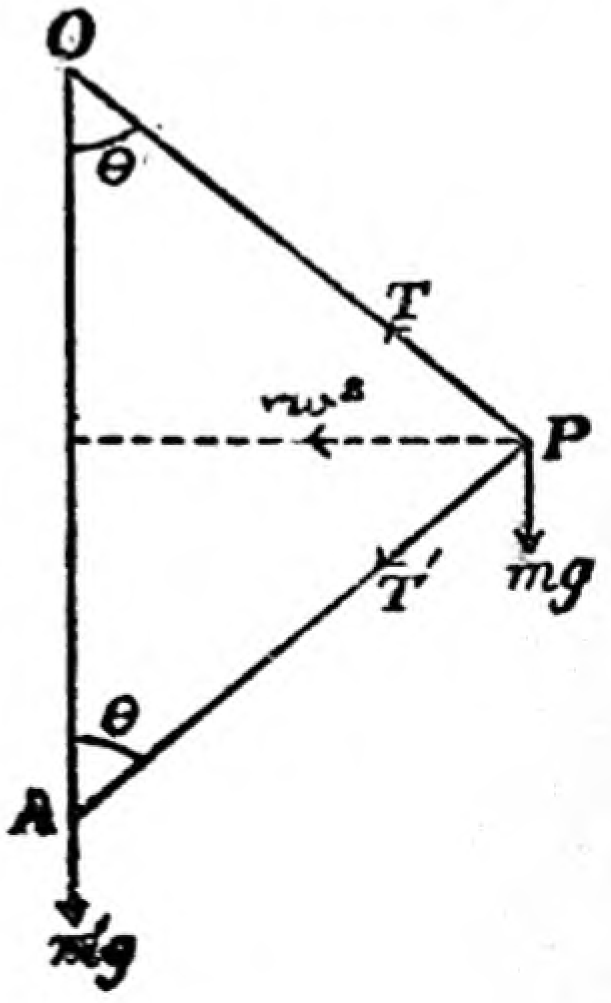
$$\therefore \quad n = 2$$

i.e., the greatest number of revolutions per second=2.

Ex. 9. *A particle, of mass m is attached to the middle of a light string of length 2l, one end of which is fastened to a fixed point and the other end to a smooth ring of mass m′ which slides on a vertical rod through O. If the particle describes a horizontal circle with uniform angular velocity ω about the rod, prove that the inclination of either portion of the string to the vertical is*

$$\frac{\cos^{-1}(m+2m')g}{ml\omega^2}.$$

Let r be the radius of the circle described by P and θ the angle which each portion of the string makes with the vertical.



$$OP = AP = l$$
$$\therefore \quad r = l \sin \theta.$$

Let T and T′ be the tensions in the portions PO and PA of the string.

Considering the motion of P, resolving horizontally and vertically,

$$T \sin \theta + T' \sin \theta = mr\omega^2 = ml \sin \theta . \omega^2$$
$$\therefore \quad T + T' = ml\omega^2 \qquad \ldots(1)$$
$$\text{and} \quad (T - T') \cos \theta = mg. \qquad \ldots(2)$$

Also, for the equilibrium of A,

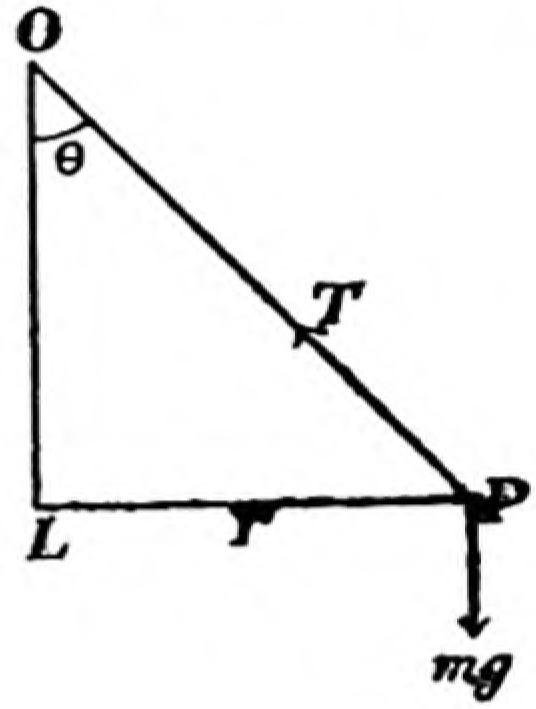
$$T' \cos \theta = m'g \qquad \ldots(3)$$

From (1), (2) and (3), we get

$$2T' \cos \theta = ml\omega^2 \cos \theta - mg$$
$$\text{or} \quad 2m'g = mlw^2 \cos \theta - mg$$
$$\cos \theta = \frac{m + 2m'}{ml\omega^2} . g.$$

Ex. 10. *A particle of mass m is attached to an elastic string of natural length a and modulus equal to the weight of the particle. Find the least angular velocity with which it can move in a conical pendulum.*

Let l be the extended length of the string, T the tension of the string and θ the angle which the string makes with the vertical.



Then $\quad \mathrm{T}\cos\theta = mg \qquad \ldots(1)$

and $\quad \mathrm{T}\sin\theta = mr\omega^2 = ml\omega^2\sin\theta$

$\therefore \quad \mathrm{T} = ml\omega^2 \qquad \ldots(2)$

By Hooke's Law,

$$\mathrm{T} = mg\,\frac{l-a}{a} \qquad \ldots(3)$$

$$\therefore \qquad mg\,\frac{l-a}{a} = ml\omega^2.$$

$$\therefore \qquad l = \frac{ag}{g-a\omega^2}.$$

Now from (1) and (2),

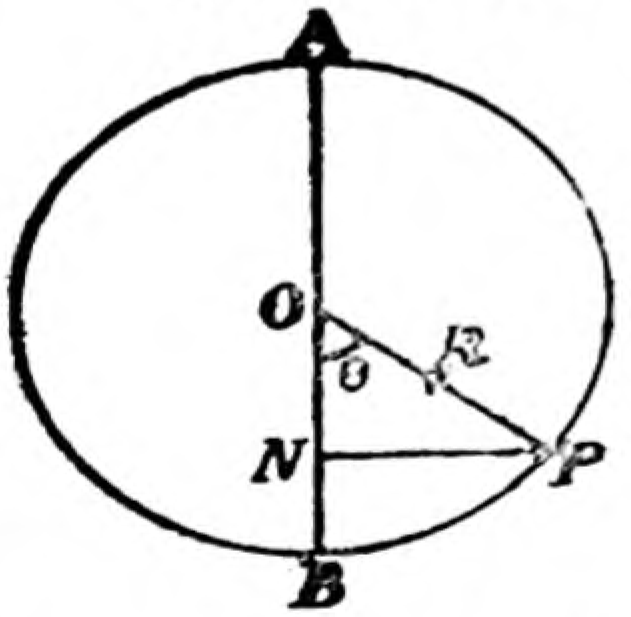
$$\cos\theta = \frac{mg}{\mathrm{T}} = \frac{g}{l\omega^2} = \frac{g-a\omega^2}{a\omega^2}.$$

Since $\cos\theta \leqslant 1$, $\therefore \quad g-a\omega^2 \leqslant a\omega^2$

or $\qquad \omega^2 \geqslant \dfrac{g}{2a} \qquad$ or $\qquad \omega \geqslant \sqrt{\dfrac{g}{2a}}.$

6·5 Rotating sphere. *A hollow smooth sphere is rotating with uniform angular velocity ω about a vertical diameter, to find the position of relative rest of a particle placed in it.*

Let a be the radius of the sphere and PN$=r$ the radius of the horizontal circle described by the particle P such that OP makes an angle θ with ON.



Let R be the pressure of the sphere on the particle.

Then $\quad \mathrm{R}\cos\theta = mg \qquad \ldots(1)$

and $\quad \mathrm{R}\sin\theta = mr\omega^2$

$\qquad = m\omega^2 a\sin\theta \qquad \ldots(2)$

$$\therefore \qquad \tan\theta = \frac{a\omega^2}{g}\sin\theta.$$

or $\qquad \sin\theta\cos\theta = \dfrac{g}{a\omega^2}\sin\theta.$

$\therefore$ either $\sin\theta = 0$, *i.e.*, $\theta = 0$ so that from (1), R$=mg$ which gives the lowest position as the position of rest.

or $\qquad \cos\theta = \dfrac{g}{a\omega^2}.$

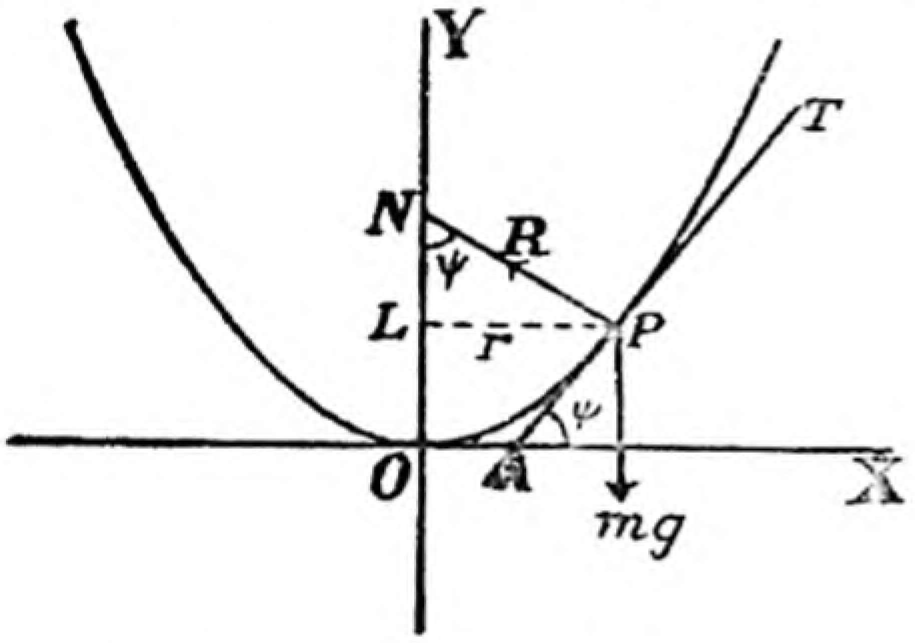
$$\because \cos\theta < 1 \quad \therefore \frac{g}{a\omega^2} < 1$$

or

$$\omega^2 > \frac{g}{a}.$$

$\therefore$ if the angular velocity is less than $\sqrt{\dfrac{g}{a}}$, the particle will rest only at the lowest point.

Ex. 11. *A smooth parabolic cup is set whirling with its axis vertical, with an angular velocity* ω. *Show that if* $\omega^2 = \dfrac{g}{2g}$, *a particle well rest in relative equilibrium on any part of its surface, where* $4a$ *is the latus rectum of the parabolic section by any plane through the axis.*



Since subnormal in a parabola is constant and equals semi-latus rectum, therefore,

$$LN = 2a \qquad \text{...(I)}$$

where PN is the normal at P and PL is $\perp$ on the axis.

$$\angle LNP = \angle TAX = \psi. \qquad \text{...(II)}$$

Let R be the reaction of the surface on the mass then because the surface is smooth, the reaction is normal, *i.e.*, along PN. Resolving vertically

$$mg = R\cos\psi. \qquad \text{...(III)}$$

Since, PL is the radius of the circle described by P,

$$M\omega^2 r = R\sin\psi \qquad \text{...(IV)}$$

$$\tan\psi = \frac{\omega^2 r}{g}$$

or

$$\frac{r}{2a} = \frac{\omega^2 r}{g}$$

or

$$\omega^2 = \frac{2a}{g},$$

which being independent of r, mass can rest anywhere.

Exercise VI (C)

1. A light chain 10 ft. long is attached at one end to a fixed point and at the other end to a small ball of mass 20 oz. If the ball revolves in a conical pendulum with an angular velocity of 176 radians per minute, find the tension in the chain to the nearest ounce.

2. If the velocity of the bob of conical pendulum is v and

the length l, prove that the inclination of the string to the vertical is given by

$$\frac{\sin^2\theta}{\cos\theta}=\frac{v^2}{gl}.$$

3. A heavy particle fastened by a light inextensible string to a fixed point A, is moving in a horizontal circle at the rate of n revolutions per sec. Prove that A is at a distance $\frac{g}{4\pi^2n^2}$ vertically above the centre of the circle. *(P.U. B.Sc. 1962)*

4. A horizontal bar AB of length a is made to rotate with a constant angular velocity ω about a vertical axis through the end B. If a particle is attached to A by a string of length l, and the string makes an angle θ with the vertical when the motion is steady; prove that

$$a\cos\theta+l\cos\theta=\frac{g}{\omega^2}. \qquad (P.U.)$$

5. A string of length l has ends fastened to two points A and B in the same vertical line at a distance a apart. If a smooth bead P on the string rotates uniformly about B so that BP is always horizontal, prove that the angular velocity is

$$l\left[\frac{2g}{a(l^2-a^2)}\right]^{\frac{1}{2}}.$$

Show also that P is nearer to B than A by

$$\frac{a^2}{\ }.$$

6. One end of a string $2l$ ft. long is fastened to a point A on a fixed vertical rod, the other to a small ring P of mass m lb. which slides on the rod; another mass Q of m' lb. is fastened to the middle point of the string and revolves with velocity v in a horizontal circle so that the angle AQP is a right angle; prove that

$$v^2=\frac{lg}{\sqrt{2}}\cdot\frac{2m+m'}{m'}. \qquad (P.U.)$$

7. Two masses M and m are connected by a string of length a which passes through a small ring. Find how many revolutions per second must m make as a conical pendulum in order that M may hang at rest at a distance b from the ring.

8. A particle of mass m is attached by a light rod of length l to a fixed point at a height $\frac{1}{2}l$ above a smooth horizontal table on which the particle is describing a circle with velocity v. Show that there will be a pressure on the table if $v^2<\frac{3}{2}gl$, and find this pressure.

9. An elastic string of natural length l is extended by an amount x when it supports a mass m at rest and is extended by an amount y when it is rotating as a conical pendulum carrying a particle of the same mass. Show that

$$gy = x(l+y)\omega^2.$$

10. A particle is tied by two equal strings each of length a to two points in the same vertical line, about which the particle revolves with constant angular velocity ω. If the strings are both taut and inclined at an angle θ to the horizontal, prove that the ratio of their tensions is

$$(a\omega^2 \sin\theta + g) : (a\omega^2 \sin\theta - g).$$

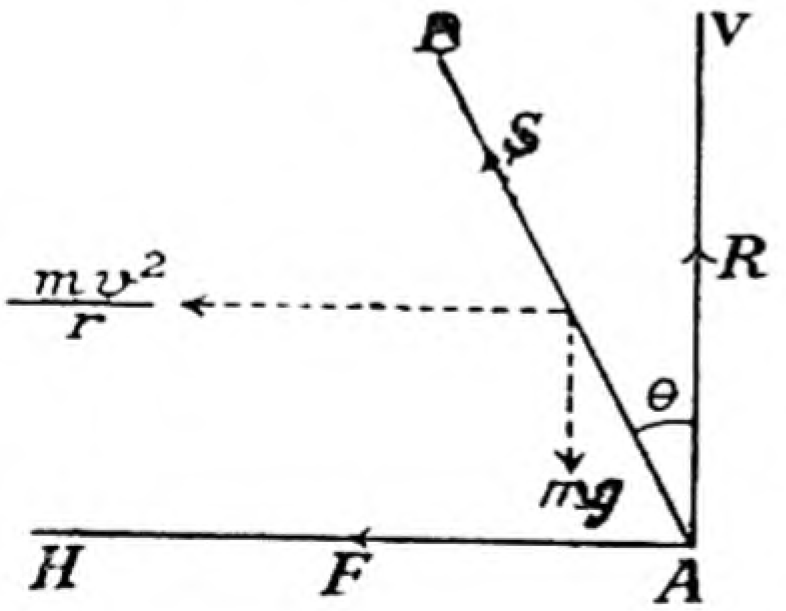
11. A conical pendulum consists of a light elastic string of natural length l and modulus equal to five times the weight of the particle. The particle is whirled so as to move in a conical pendulum of height l. Show that when the motion is steady, the velocity of the particle must be $\frac{3}{4}\sqrt{gl}$. *(P.U.)*

6·6 Motion of a cyclist on a curved (nearly circular) track.

When a cyclist goes round a circular path, the necessary inward normal force $\frac{mv^2}{r}$ is supplied by the force of friction between tyres and the ground (towards H).

The reader must have experienced or observed that the cyclist leans towards the centre of the path so that the reaction of the ground becomes inclined to the vertical.

The cycle in the inclined position making an angle θ with AV, the vertical, is shown by AB, with the weight of the machine and the rider acting at the centre of gravity G.



Let S be the resultant reaction, *i.e.*, the resultant of the normal reactions R and F, the limiting force of friction (because the Dynamical friction is equal to limiting friction practically).

Then resolving S horizontally and vertically

$$S\sin\theta = \frac{mv^2}{r} = F \qquad \ldots \text{(I)}$$

$$S\cos\theta = mg = R \qquad \ldots \text{(II)}$$

so that

$$\tan\theta = \frac{v^2}{gr} \qquad \ldots\text{(III)}$$

Thus $\tan\theta$ and consequently θ is large if v is large or r is small, *i.e.*, the cyclist is in danger of falling while taking a fast turn or sharp turn and this fact can be verified from every day experience.

Further if μ denotes the coefficient of friction and λ the angle of friction, then

$$F=\mu R=R \tan \lambda,$$

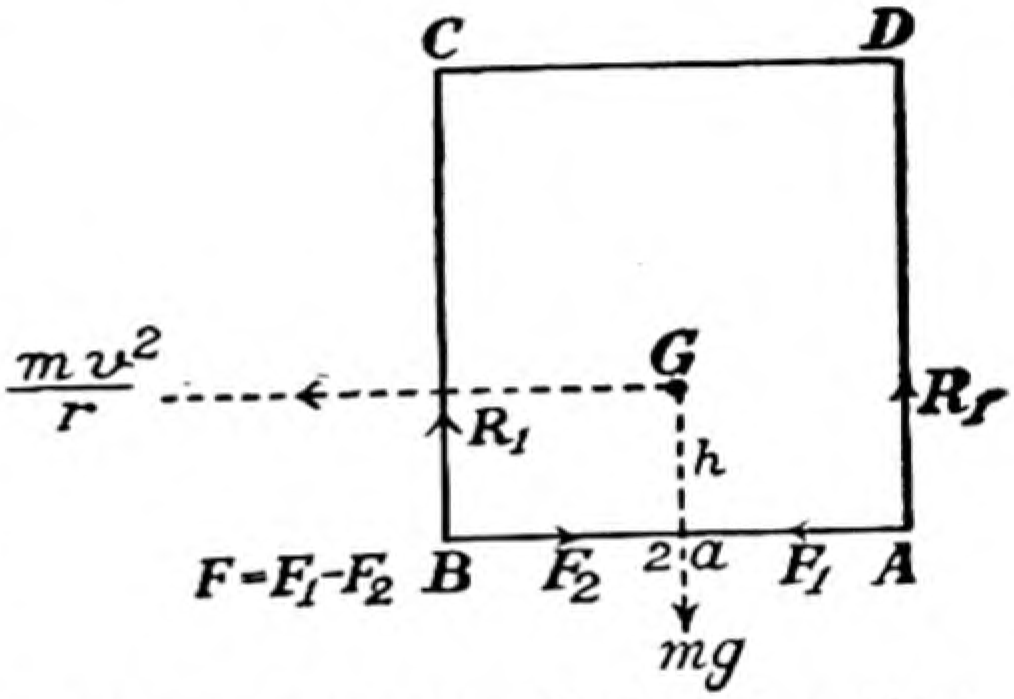
so that
$$\tan \lambda=\frac{F}{R}=\frac{S \sin \theta}{S \cos \theta}=\tan \theta$$

i.e., θ the inclination of the cyclist to the vertical when the friction is limiting is equal to the angle of friction between the ground and the wheels.

6·7 Motion of a Railway Carriage or a Car on a Curved (nearly Circular) Track.

In this case, the necessary normal force $\frac{mv^2}{r}$, acting at the centre of gravity is supplied by the horizontal component of the pressure of the rails on the flanges of the wheels.

Consider a normal section ABCD, of the carriage by the plane perpendicular to the direction of motion and let G be the centre of gravity of this section and A and B the points of contact of the wheels and the rails.



Let R_1 and R_2 be the vertical components and F_1 and F_2 the horizontal components of the reaction of the rails at A and B and these supply the effective force $\frac{mv^2}{r}$ towards the centre of the path. Resolving in the horizontal and vertical directions, we get

$$F_1-F_2=\frac{mv^2}{r} \qquad \text{... (I)}$$

$$R_1+R_2=mg \qquad \text{... (II)}$$

Taking moments about A and B, separately, we have

$$\frac{mv^2}{r}h=R_1.2a-mga$$

where $AB=2a$ and h is the height of G above the ground.

$$\therefore \qquad R_1=\tfrac{1}{2}m\left(g+\frac{v^2h}{ar}\right) \qquad \text{...(III)}$$

and
$$R_2=\tfrac{1}{2}m\left(g-\frac{v^2h}{ar}\right) \qquad \text{...(IV)}$$

From these results we conclude that R_1 is always positive and increases with v, while R_2 decreases and when $v^2=\frac{gar}{h}$, R_2 vanishes which shows that the wheels on the inside of the curve are no longer in contact with the rails and the railway carriage will upset if

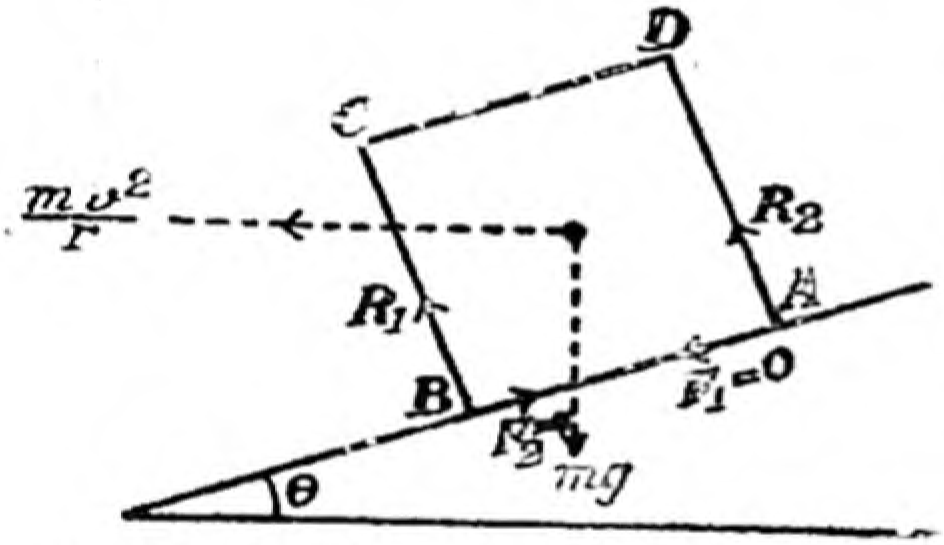
$$v > \sqrt{\frac{gar}{h}}. \qquad \ldots \text{(V)}$$

Clearly v is large, when a is large and h is small ; so that the danger of upsetting is minimized by keeping the centre of gravity as low as possible and the distance between the wheels sufficiently large.

Quite similar is the discussion of motion of a car in a circular track.

6·71 Motion on a banked-up track.

In the case of a railway carriage or a car moving on a curved level track, the forces F_1 and F_2 act on the wheels in the horizontal direction. Now action and reaction being equal and opposite, the wheels also exert equal amount of pressure on the rails in the opposite direction. These lateral pressures are very harmful in as far as they wear away the rails and the wheels and may also tend to upset the carriage. In order to avoid these lateral pressures, the outer rail is raised up, *i.e.*, the track is banked-up so that the floor of the carriage is not horizontal.



Let θ be the inclination of AB to the horizontal and R_1 and R_2 be the normal reaction perpendicular to AB. Resolving horizontally and vertically,

$$(R_1+R_2)\sin\theta = \frac{mv^2}{r}$$

$$(R_1+R_2)\cos\theta = mg$$

$$\therefore \qquad \tan\theta = \frac{v^2}{rg} \quad \text{or} \quad \theta = \tan^{-1}\frac{v^2}{gr} \qquad \ldots \text{(I)}$$

which gives us the angle of banking up for a given v and r. For this value of θ, lateral thrust is absent.

6·72 In banking up the railway tracks and roads, the angle θ is adjusted for some moderate velocity with which the vehicles are expected to run. If the actual velocity V of the vehicle is different from v for which θ has been adjusted, some lateral thrust F comes into play.

Let F act along AB, when the velocity of the carriage is V (different from the velocity v)

$$\text{Then} \qquad \frac{mV^2}{r}\cos\theta = F + mg\sin\theta$$

$$\therefore \qquad F=\frac{mV^2}{r}\cos\theta-mg\sin\theta$$

$$=m\cos\theta\left(\frac{V^2}{r}-g\tan\theta\right)$$

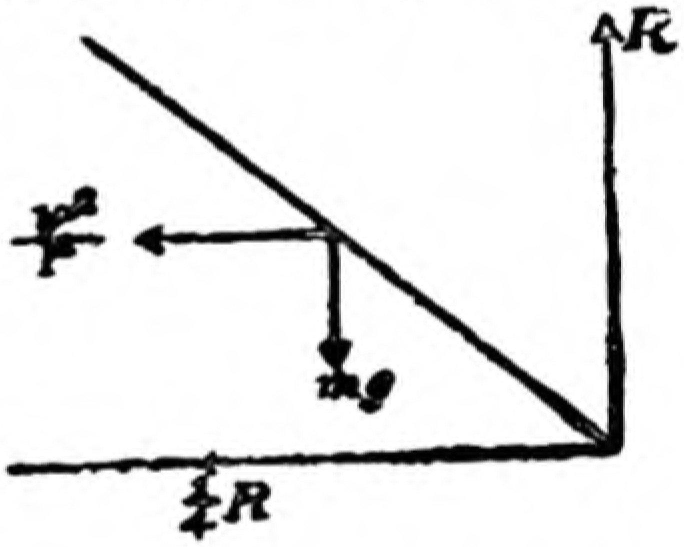
$$=m\cos\theta\left(\frac{V^2-v^2}{r}\right),\text{ using the above (I)}$$

Case I. If $V>v$, $F>0$, so that the lateral thrust acts along AB and is caused by the outer rail at A (by outer rim of the road in case of a car).

Case II. If $V<v$, $F<0$, *i.e.*, the thrust acts along BA and is caused by the inner rail.

Ex. 12. *A cyclist riding at 16 ft./sec. has to turn a corner. What is the least radius of the curve he has to describe if the coefficient of friction between the tyres and the road be ¼ ?*

Let R be the normal reaction, m the mass of the bicycle and the man, r the radius of the circle described when the friction is maximum.



Force of friction $=\frac{1}{4}R$

$$\therefore \qquad \tfrac{1}{4}R=m\,\frac{v^2}{r}$$

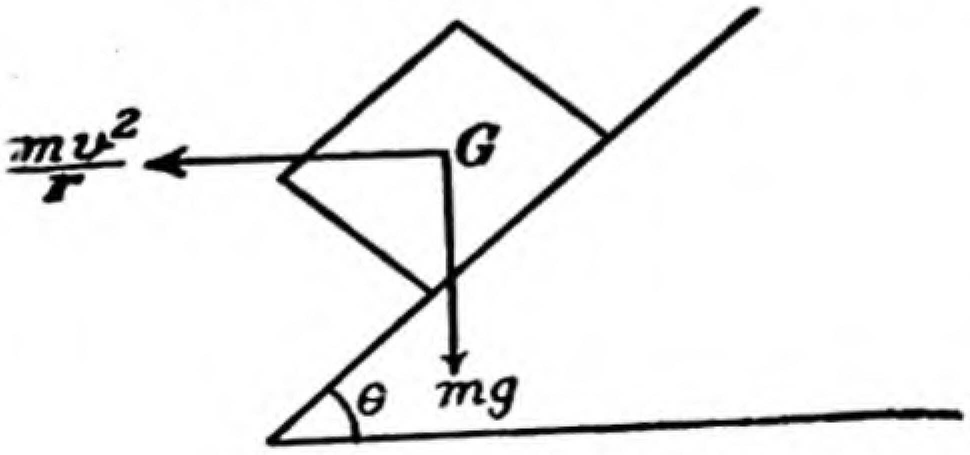
$$\text{But} \qquad R=mg$$

$$\therefore \qquad \tfrac{1}{4}mg=m.\frac{(16)^2}{r}$$

$$\therefore \qquad r=32\text{ ft.}$$

Ex. 13. *A train is running at 45 m.p.h. on a curve of mean radius 1200 ft. and the distance between the rails is 4 ft. 8½ inches. Find how much the outer rail must be raised in order that there may be no side thrust on the rail.*

Let m lb. be the mass of the train and θ the angle the plane of the rails makes with the horizontal.



$$v=45\text{ m.p.h.}=66\text{ ft./sec.}$$

$$\text{Now } mg\sin\theta=\frac{mv^2}{r}\cos\theta$$

$$\therefore \qquad \tan\theta=\frac{v^2}{g\,.\,r}=\frac{66\times 66}{32\times 1200}$$

$$=\frac{363}{3200}$$

The height to which the outer rail should be raised

$$=(4\text{ ft. }8\tfrac{1}{2}\text{ inches})\sin\theta$$

$$=56\tfrac{1}{2}\sin\theta\text{ inches}$$

Since θ is small, we can take $\sin\theta=\tan\theta$.

$$\therefore \quad \text{height required} = 56\tfrac{1}{2} \times \frac{363}{3200} = \frac{113}{2} \times \frac{363}{3200} = 6 \cdot 4 \text{ inches.}$$

Exercise VI (D)

1. A train is moving at the rate of 30 m.p.h. on a level track of radius 1200 ft. If the distance between the rails be 6 ft. find how much the outer rail must be raised above the inner so that there may be no lateral thrust on the rails ? *(P.U., 1954)*

2. A cyclist riding at $7\frac{1}{2}$ miles per hour, has to turn a corner. What is the least radius of the curve he has to describe, if the coefficient of friction between the tyres and the road is 0·3 ?

3. A cyclist travels a level circular track of radius 220 feet, and the coefficient of friction between the tyres and the ground is 0·32. Find the greatest speed at which he may travel.

(P.U., 1953 S.)

4. The shape of a motor-cycle track at a corner is that of a circle whose radius is 100 yards. Find the angle at which the track should be inclined to the horizontal in order that a rider can take the corner at 30 m.p.h. without any lateral reaction between his motor-cycle and the track. If a motor-cyclist can take the corner safely at 60 m.p.h., find the least possible value of the coefficient of friction between the track and his tyres.

5. A motor track describes a curve of 250 ft. radius, and is sloping downwards towards the inside of the curve at an angle $\tan^{-1}\frac{1}{5}$. At what speed must a car turn along it so that there should be no tendency to side-slip ?

6. Prove that the radius of the sharpest curve on which a motor car can turn without tilting at a speed of V ft./sec. is $\dfrac{2V^2 h}{ga}$; when h is the height of the centre of gravity and a is the width of the track between the wheels.

7. An engine of weight W is at rest on a railway curve of radius r which is banked up to suit a speed v ; show that there is an internal lateral thrust on the rail of magnitude

$$\frac{W v^2}{\sqrt{v^4+g^2r^2}}.$$

8. If the maximum and minimum speeds of trains and a track of radius r are u and v respectively, show that if the outer rail is banked up at angle β, where

$$\tan\beta = \frac{v^2+u^2}{2gr},$$

the outer lateral thrust of the fastest train is equal to the inner lateral thrust of the slowest one assuming their weights are equal.

9. If a particle of weight W is suspended from the roof of a railway carriage which is moving with speed v in a horizontal circular track of radius r. Prove that in a position of relative rest the tension of the string is

$$W\left(1+\frac{v^4}{g^2r^2}\right)^{\frac{1}{2}}.$$

10. A certain curved portion of Kangra Valley railway track with gauge $2\frac{1}{2}$ ft. has a radius of 242 ft. and super-elevation of the outer rail is $1\frac{1}{5}$ inches. Find the velocity with which a train can travel without wearing away the rails.

11. In Q. 10, what is the lateral thrust on the track produced by an engine of weight 32 tons

(*i*) moving at a speed of 15 m.p.h. ?
(*ii*) moving at a speed of 10 m.p.h. ?
(*iii*) at rest ?

ANSWERS

Exercise VI (A), Page 132

2. $62\frac{1}{2}$ rev. ; 50 sec. **3.** 30 r.p.s. ; 4 min.

7. $\dfrac{agt}{a^2+\frac{1}{4}g^2t^4}$. **8.** $\dfrac{v\sqrt{x^2-a^2}}{ax}$.

Exercise VI (B), Page 138

2. (*b*) Vel. $=a\omega\sin\omega t$, when a is the radius.
3. Accel. $=784\pi^2$, $T=49\pi^2$ lbs. wt. **4.** 18·3 lbs. wt.

6. $\dfrac{1}{2\pi}\sqrt{\dfrac{Mg}{ml}}$. **8.** $\sqrt{2cg}$.

9. $\dfrac{480}{\pi}$ r.p.m. **10.** $\sqrt{\dfrac{r\lambda\ (r-l)}{lm}}$.

11. $2\pi\sqrt{\dfrac{l}{g}}$.

12. $V=1280\sqrt{330}$ f.p.s. ; $T=125\pi\sqrt{330}$ sec.

Exercise VI (C), Page 144

1. 3 lb. 6 oz. nearly. **7.** $\dfrac{1}{2\pi}\sqrt{\dfrac{Mg}{m(a-b)}}$.

8. $m\left(g-\dfrac{2v^2}{3l}\right)$.

Exercise VI, (D) Page 150

1. 3·63 inches. **2.** $12\frac{29}{48}$ ft. **3.** $\frac{32}{5}\sqrt{55}$ f.p.s.
4. 11° 24′, 0·52. **5.** 40 f.p.s. **10.** 12 m.p.h.

11. (*i*) $\dfrac{18}{\sqrt{626}}$ tons wt. (*ii*) $\dfrac{88}{9\sqrt{626}}$ tons wt.

14. $\dfrac{32}{\sqrt{626}}$ tons wt.

CHAPTER VII

MOTION IN A SMOOTH VERTICAL CIRCLE

Simple Pendulum

7·1 Equations of Motion for a Circular Path.

In Art. 6·33, it was established that whenever a particle moves in a plane circle, the impressed forces acting on it may be resolved in two perpendicular directions and equated to

(*i*) effective force along the tangent $\left(mv\frac{dv}{ds}\right)$

(*ii*) effective force along inward normal $\left(\frac{mv^2}{r} \text{ or } mw^2r\right)$

thus giving us two equations of motion.

As explained earlier also in Art. 6·31 ; the tangential forces change the magnitude of the velocity while the normal forces bring a change in the direction of motion.

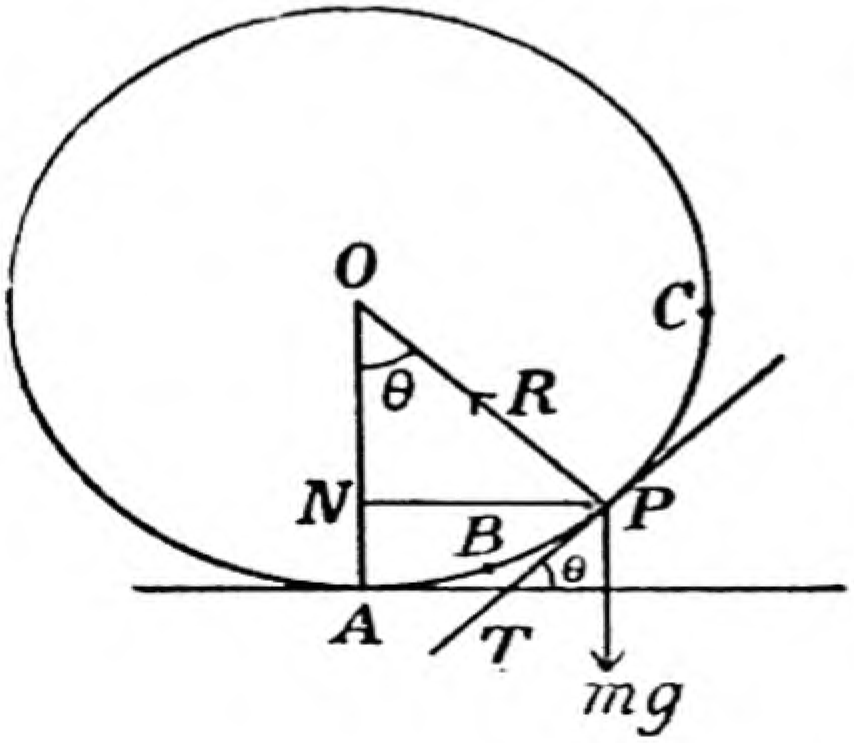
7·2. Motion in a smooth vertical circle under gravity alone.

Since the tangential force affects the magnitude of velocity alone, therefore in all problems pertaining to motion in a vertical circle, it will be found advisable to discover, once for all, an expression for the magnitude of velocity and as it shall be seen below (Art. 7·21), such an expression is obtainable from the Tangential equation of motion without any reference to the Normal equation.

7·21 *A particle is projected along the inside of the arc of a smooth vertical circle from any point on it with velocity* v_1. *To prove that* v_2, *the velocity after it has moved up a vertical height* h, *is given by*

$$v_2^2 = v_1^2 - 2gh. \qquad \textit{(P.U. 1968)}$$

Let at any time t, P be the position of the particle moving inside the smooth vertical circle of radius r and centre O.



Let A be the lowest point on the circle, B the point of projection where the velocity is v_1 and C the position of the particle where its velocity is v_2.

Let $\angle$AOP$=\theta$ and arc AP$=s$.

Then $s = r\theta$. ...(I)

The only forces acting on the particle are (*i*) its weight *mg* acting vertically downwards and (*ii*) the pressure R of the circle in the direction of normal.

If v is the velocity at P, the equation of motion along the tangent at P is

$$mv\frac{dv}{ds}=-mg\sin\theta \qquad \ldots\text{(II)}$$

Now
$$\frac{dv}{ds}=\frac{dv}{d\theta}\cdot\frac{d\theta}{ds}$$
$$=\frac{1}{r}\frac{dv}{d\theta}, \qquad \text{by (I)}$$

$\therefore$
$$v.\frac{1}{r}\frac{dv}{d\theta}=-g\sin\theta$$

or
$$v\frac{dv}{d\theta}=-gr\sin\theta.$$

Integrating, we get

$\frac{1}{2}v^2=gr\cos\theta+C$, where C is a constant of integration.

$$=g.\text{NO}+C$$
$$=g(\text{AO}-\text{AN})+C$$
$$=g(r-y)+C,$$

where y is the height of P above A.

Let $y=y_1$ for B and $y=y_2$ for C.

At B, $v=v_1$ and $y=y_1$

$\therefore$ $\frac{1}{2}v_1^2=g(r-y_1)+C$ ...(III)

At C, $v=v_2$ and $y=y_2$

$\therefore$ $\frac{1}{2}v_2^2=g(r-y_2)+C$ ...(IV)

From (III) and (IV), by subtraction,

$$\frac{1}{2}v_2^2-\frac{1}{2}v_1^2=-g(y_2-y_1)$$

$=-gh$ where h is the height of C above B.

$\therefore$ $v_2^2=v_1^2-2gh.$ ...(V)

7·22 Similarly we may prove that the velocity v_2 attained by the particle when it is projected and moves down the arc of the circle through a vertical depth h, is given by

$$v_2^2=v_1^2+2gh. \qquad \ldots\text{(VI)}$$

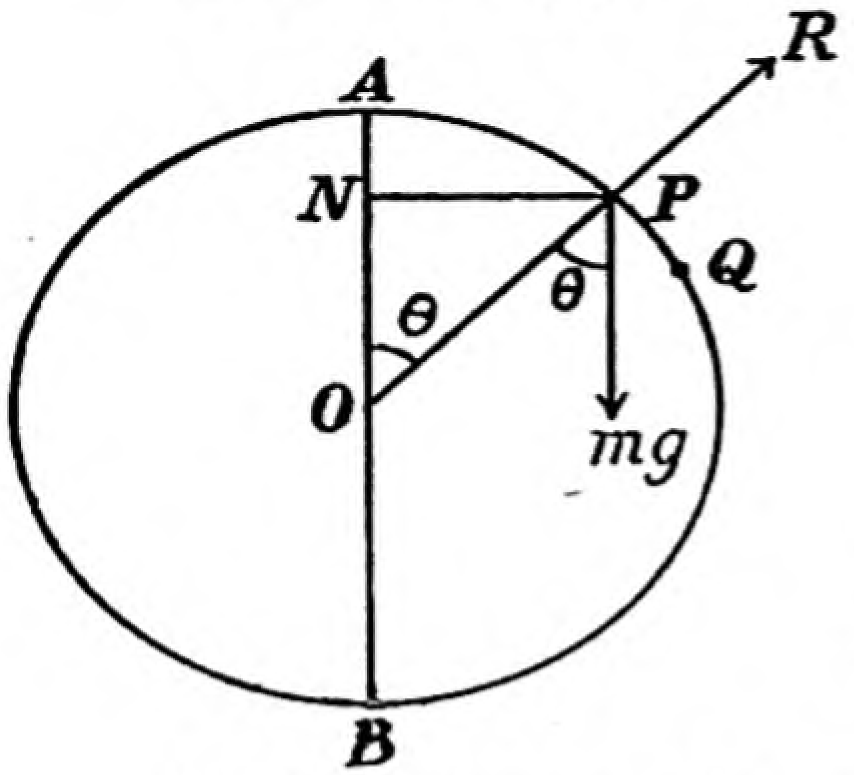
Ex. A particle is projected from the lowest point with velocity u and moves along the inside of a smooth vertical circle of radius. Prove that the velocity v of the particle at a vertical distance h from the lowest point is given by

$$v^2=u^2-2gh.$$ *(P.U. B.Sc., 1961)*

7·3 Motion on the outside of a smooth vertical circle.

A particle starts from rest at the highest point and moves down along the outside of the arc of a smooth vertical circle, to prove that the particle leaves the arc when it is at a vertical distance of one-third of the radius of the circle from the highest point. (Delhi, 1962 53)

Let P be any position of the particle moving outside the arc of the vertical circle of radius r. Let PN be perpendicular to AB, the vertical diameter of the circle.



Let $\angle AOP=\theta$ and $AN=h$.

If v is the velocity at P, then by (VI) of Art. 7·22

$$v^2=2gh.$$

The external forces acting on the particle are (*i*) its weight mg acting vertically downwards and (*ii*) reaction R of the curve outwards along OP as the particle is pressing the arc inwardly.

Writing the equation of motion along the normal PO, we get

$$\frac{mv^2}{r}=mg\cos\theta-R$$

$$\therefore \quad R=m\left(g\cos\theta-\frac{v^2}{r}\right)$$

$$=m\left(g\cdot\frac{r-h}{r}-\frac{2gh}{r}\right)$$

$$=mg\,\frac{r-3h}{r}.$$

The particle leaves the circle when

$$R=0$$

i.e., when $r=3h$

or $h=\dfrac{r}{3}$. ...(I)

$$\therefore \quad ON=\frac{2r}{3}$$

$$\therefore \quad \cos\theta=\frac{ON}{OP}=\frac{2}{3}.$$

Ex. A particle of mass m, placed on top of a smooth vertical circle is slightly displaced, show that the particle will leave the circle when the radius to the particle makes an angle θ with the upward vertical given by $\cos\theta=\frac{2}{3}$ and the velocity then is $\frac{\sqrt{2ag}}{3}$.

(*J. & K.*)

8·31 Subsequent Path.

After leaving the circle the particle moves in a parabolic path at the point Q.

Suppose $\angle AOQ=\theta$, then $\cos\theta=\dfrac{ON}{OQ}=\dfrac{\frac{2}{3}r}{r}=\dfrac{2}{3}$ ...(II)

Let v be the velocity of the particle when it leaves the curve along the tangent at that point.

Then $$v^2=2gh=2g\,.\,\frac{r}{3}=\frac{2gr}{3}. \qquad \text{...(III)}$$

The equation of the trajectory referred to the horizontal line through the point where the particle leaves the curve (taken as the origin) as the x-axis and the *downward* vertical through the origin as the y-axis is

$$y=x\tan\theta+\frac{gx^2}{2v^2\cos^2\theta}$$

$$=\frac{\sqrt5}{2}x+\frac{gx^2}{2\,.\,\frac{2}{3}gr\,.\,\frac{4}{9}}$$

$$\left(\because \quad \cos\theta=\frac{2}{3}, \quad \therefore \quad \tan\theta=\frac{\sqrt5}{2}\right)$$

or $$y=\frac{\sqrt5}{2}x+\frac{27}{16r}x^2 \qquad \text{...(V)}$$

The *latus rectum* of this parabola is $\frac{16r}{27}$ (the reciprocal of the coefficient x^2).

Exercise VII (A)

1. (*a*) A particle starts from rest at the highest point and moves down along the outside of the arc of a smooth vertical circle. Find the angle θ which the radius vector to the particle makes with the vertical when it leaves the circle.

(*b*) Find θ when (*i*) it starts with a velocity u at the highest point ; (*ii*) it starts from rest from a point where $\theta=\alpha$.

(*c*) Find also the latus rectum of the subsequent parabolic path in each case. (*P.U.*)

2. Prove that in Q. 1 above, if the particle is initially projected with a tangential velocity of $\sqrt{\frac{1}{2}gr(3\sqrt2-4)}$, from the highest point, it will leave the circle when the radius to the particle makes angle of 45° with the upward vertical. (*P.U.*)

3. A particle slides from rest down the rim of a smooth vertical circular disc, starting from the position in which the radius through the particle makes an angle α with the vertical. Prove that the particle leaves the disc when the radius through it makes an angle $\cos^{-1}(\frac{2}{3}\cos\alpha)$ with the vertical.

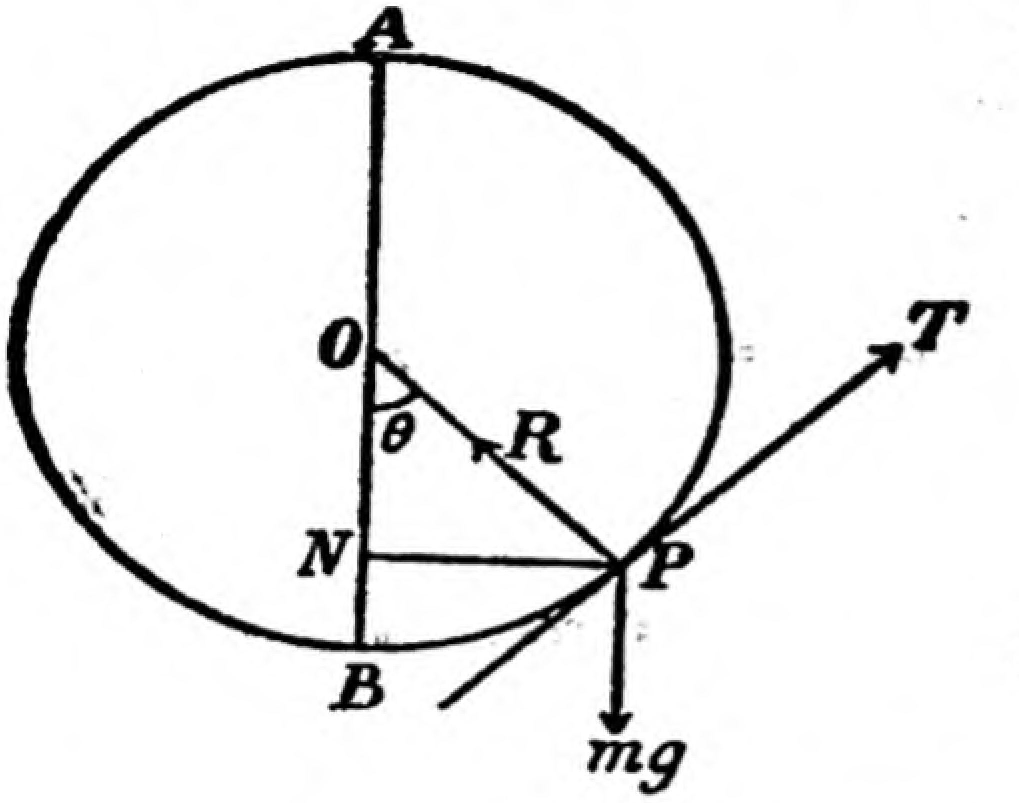
If the particle starts from rest at a depth $\frac{1}{2}r$ below the highest point, show that it leaves the disc at a height of $\frac{1}{3}r$ above the centre.

4. A particle slides down the surface of a smooth fixed sphere of radius a, being slightly displaced from rest at the highest point. Find where it will leave the sphere, and show that it will afterwards describe a parabola of latus-rectum $\frac{16}{27}a$. and that it will strike the horizontal plane through the lowest point of the sphere at a distance $\frac{5(\sqrt5+4\sqrt2)a}{27}$ from the vertical diameter. (*P.U. 1960 S ; 1961 S*)

7·4 Motion on the inside of a smooth vertical circle.

A particle is projected from the lowest point with velocity u and moves along the inside of the arc of a smooth vertical circle, to discuss the motion. (*Delhi, 1954*)

Let P be the position of the particle at time t. Draw PN perpendicular to AB, the vertical diameter of the circle of radius r.



Let $\angle BOP=\theta$ and $BN=h$.

Then $BN=h=r-r\cos\theta$

$$\therefore \quad \cos\theta=\frac{r-h}{r}.$$

Let v be the velocity at P.

Then $$v^2=u^2-2gh \quad \ldots(1)$$

The equation of motion along the normal PO is

$$\frac{mv^2}{r}=R-mg\cos\theta$$

$$\therefore \quad R=\frac{mv^2}{r}+mg\cos\theta$$

$$=m\left[\frac{u^2-2gh}{r}+g\,\frac{r-h}{r}\right]$$

$$=\frac{m}{r}(u^2-3gh+gr). \quad \ldots(2)$$

Equations (1) and (2) give the velocity and the reaction of the curve when the particle is at a height h.

At the highest point, $h=2r$

$$\therefore \quad R=\frac{m}{r}(u^2-5gr)$$

and $$v^2=u^2-4gr.$$

At the lowest point, $h=0$

$$\therefore \quad R=\frac{mu^2}{r}+mg \quad \ldots(3)$$

Case I. The particle will describe complete circles if R and v do not vanish at the highest point

i.e. if $$u^2>5gr \quad \ldots(4)$$

$\therefore$ *The least velocity with which the particle must be projected from the lowest point so as to make complete revolutions is* $\sqrt{5gr}$. (*Delhi, 1961*)

From (3), putting $u^2=5gr$

$$R=6mg \quad \ldots(5)$$

i.e. if the particle is projected from the lowest point with velocity just sufficient to carry it to the highest point, then the pressure at the lowest point is six times the weight of the particle.

Other Cases :

Let v vanish at a height h_1 and R vanish at a height h_2.

Then from (1),
$$h_1 = \frac{u^2}{2g} \qquad \ldots(6)$$

and from (2),
$$h_2 = \frac{u^2+gr}{3g} \qquad \ldots(7)$$

Case II. The particle comes to rest if v vanishes before R vanishes.

i.e. if
$$h_1 < h_2$$

or if
$$\frac{u^2}{2g} < \frac{u^2+gr}{3g}$$

or if
$$u^2 < 2gr.$$

Now $\sqrt{2gr}$ is the velocity at the lowest point just sufficient to take the particle to the level of the horizontal diameter or centre level.

Thus if the velocity of projection u is less than $\sqrt{2gr}$, the particle stops instantaneously at a point lower than the horizontal diameter. The particle will then begin to move down. It reaches the lowest point with velocity u and then rises to an equal height to the other side of B.

Case III. The particle leaves the curve (R=0) before the velocity vanishes if
$$h_2 < h_1$$

or if
$$u^2 > 2gr.$$

i.e. if $u^2 > 2gr$ but $< 5gr$, the particle leaves the arc of the circle somewhere above the horizontal diameter but below the highest point and moves in a parabolic path.

Hence we obtain the following *three cases* :

Case I. When $u^2 > 5gr$, the particle makes complete revolutions. *(P.U. 1959 S)*

Case II. When $u^2 < 2gr$, the particle oscillates on each side of the lowest point as in Simple Pendulum. *(P.U. 1959 S)*

Case III. When $2gr < u^2 < 5gr$, it leaves the arc and moves in a parabolic path.

7·5 Motion of a bead on a smooth circular wire.

The case of a bead on a smooth circular wire or of a particle moving inside a smooth circular tube is slightly different as here the question of the particle leaving the wire or the tube does not arise. Hence two cases arise : either the body makes complete oscillations or it oscillates about the lowest point. The body will make complete oscillations if $u^2 > 4gr$. But when $u^2 > 2gr$ and $< 5gr$, the particle comes to rest for an instant at a point above the level of the horizontal diameter. The reaction now changes sign so that instead of the body pressing the wire or the tube outwards, it begins to press it inwards.

7·6 Motion of a heavy particle attached to a string in a vertical plane. *(P.U. B.Sc., 1961 S)*

The case of a particle hanging from a fixed point by a light inextensible string and projected with a certain horizontal velocity is exactly the same as that of a particle moving inside a smooth vertical circle with a difference that we have only to consider the tension T instead of the pressure R, and the length l for the radius r of the circle.

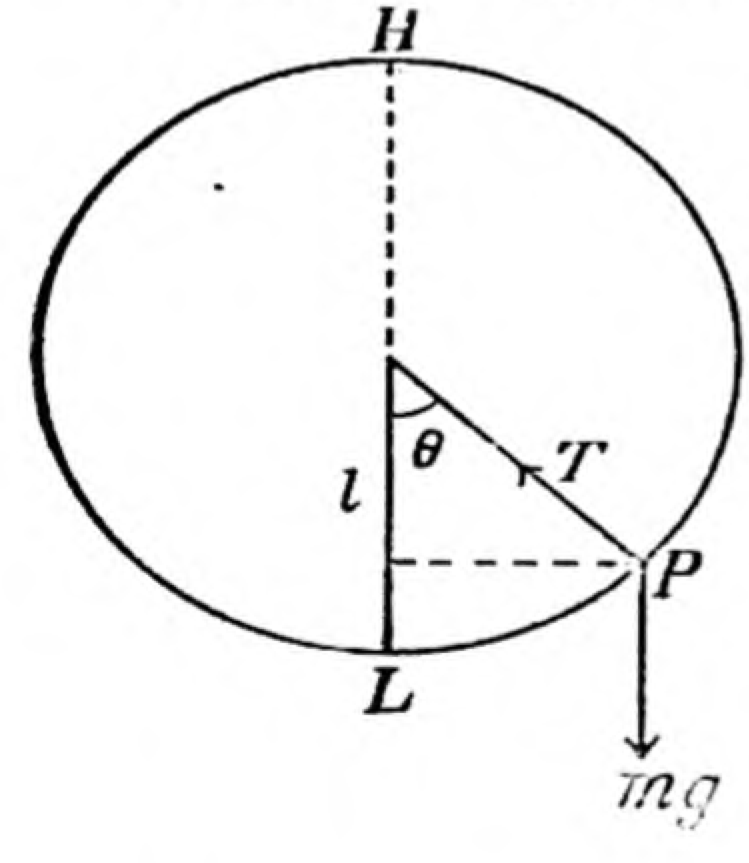
$$v^2=u^2-2gh \qquad \text{...(I)}$$

and

$$\text{T}=\frac{m}{l}\left[u^2-3gh+gl\right] \qquad \text{...(II)}$$

$$\text{T at L}=\frac{m}{l}\left(u^2+gl\right) \qquad \text{...(III)}$$

If $2gl<u^2<5gl$, the tension vanishes at a point above the point of suspension, the string is no longer tight, and the particle moves freely in a parabolic path so long as the string does not become tight again.



If $u^2 \geqslant 5gr$, then the particle will make complete revolutions, and the string should bear the tension of at least six times the weight of the particle, because in case when the particle can make complete revolutions, the tension of the string when the particle is at the lowest point (L) is six times its weight. As the particle rises higher and higher the tension decreases, till at the highest point H it instantaneously vanishes to increase further beyond H.

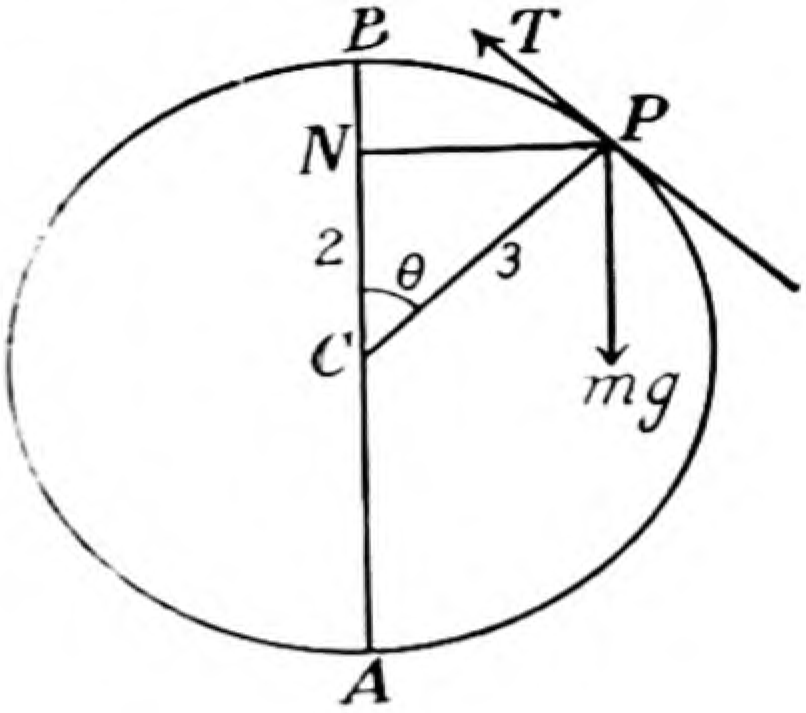
7·61 Simple Pendulum

If a heavy particle is tied to one end of a light inextensible string, the other end of which is fixed, and the particle oscillates in a vertical circle, then the system is called a *Simple Pendulum.* In Art. 7·6 above we have thus discussed the motion of a simple pendulum.

SOLVED EXAMPLES

Ex. 1. *A particle is hanging from a fixed point by a light chord 3 feet long, and is started moving with an initial horizontal speed such that the cord slackens when the particle is 5 feet above its lowest point. Find how much higher it will rise.* (*P.U. B.Sc., 1958*)

Let A be the lowest point and C the point of suspension.



Let the cord become slack in the position CP where $\angle BOP=\theta$.

Then if PN is perpendicular on AB, NA=5 ft.

$\therefore$ CN=2 ft. $\therefore$ $\cos\theta=\frac{2}{3}$.

Let v be the velocity at P. In the position when the cord slackens.

$$\frac{mv^2}{3}=mg\cos\theta \quad (\because\ T=0)$$

$$\therefore \quad v^2=3g\cos\theta$$

$$=2g.$$

$$\therefore \quad v=\sqrt{2g}=8 \text{ ft./sec.}$$

The particle now moves along the tangent PT in a free parabolic path, its initial velocity being 8 ft./sec. inclined at an angle $\cos^{-1}\frac{2}{3}$ to the horizon.

Vertical component of initial velocity $=8\sin\theta=\frac{8}{3}\sqrt{5}$ ft./sec.

Since in a trajectory, at the highest point, the direction of velocity is along the horizontal, the vertical component of the velocity at the highest point is zero.

$\therefore$ if h if the greatest height reached by the particle,

$$0=(\tfrac{8}{3}\sqrt{5})^2-2gh$$

$$\therefore \quad h=\tfrac{5}{9} \text{ ft.}$$

Ex. 2. *Show that the greatest angle through which a person can oscillate on a swing, the ropes of which can support twice the person's weight when at rest is 120°.* (*P.U. 1957 S*)

If the ropes are strong enough and he can swing through 180° and if v is his speed at any point, prove that the tension in the rope at that point is $\frac{3mv^2}{2l}$, *where m is the mass of the person and l the length of the rope.*

Let u be the initial velocity and v be the velocity in the position P when the ropes are inclined at an angle θ to the vertical.

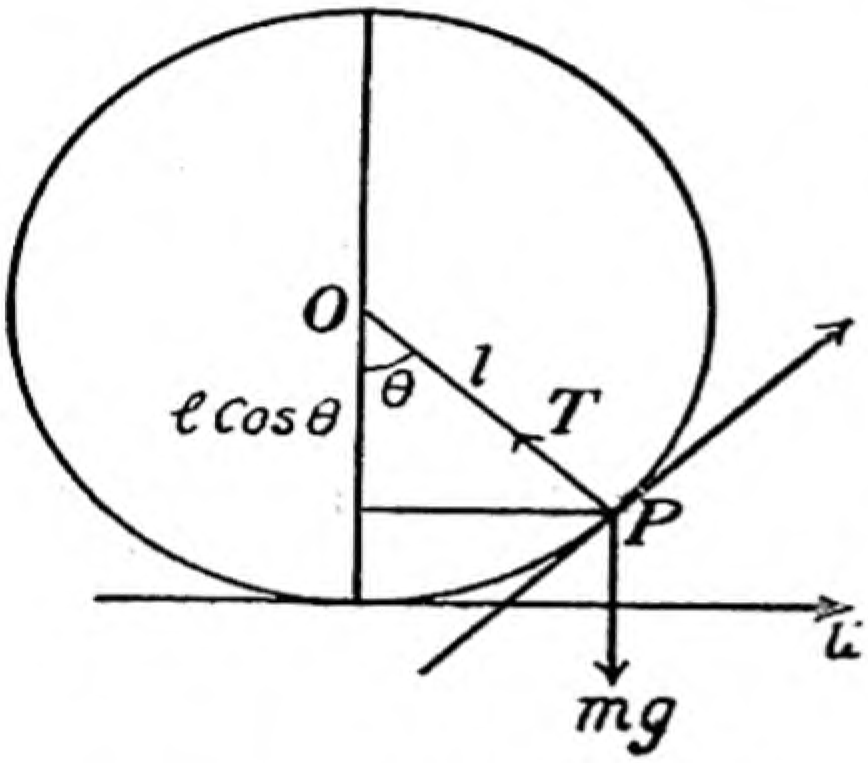
Then $$v^2=u^2-2gl(1-\cos\theta) \qquad \ldots(1)$$

The equation of motion along the normal at P is

$$\frac{mv^2}{l}=T-mg\cos\theta$$

or
$$T=m\left[g\cos\theta+\frac{v^2}{l}\right]$$
$$=m\left[g\cos\theta+\frac{u^2}{l}-2g\left(1-\cos\theta\right)\right]$$
i.e.,
$$T=m\left[\frac{u^2}{l}-2g+3g\cos\theta\right]. \qquad \ldots(2)$$

Putting $T=2mg$ and $\theta=0$ in (2), we get



$$2mg=m\left[\frac{u^2}{l}-2g+3g\right]$$

which gives $u^2=gl$,

Putting $v=0$ and $u^2=gl$ in (1), we get
$$0=gl-2gl(1-\cos\theta).$$
$$\therefore \quad \cos\theta=\tfrac{1}{2}$$
or
$$\theta=60°.$$

This gives the angle on one side of the vertical through which man can oscillate.

$\therefore$ the greatest angle through which the person will oscillate $=2\theta=120°$.

(*ii*) Again when $2\theta=180°$ or $\theta=90°$, then $v=0$.

$\therefore$ from (1), $0=u^2-2gl$

or
$$u^2=2gl. \qquad \ldots(3)$$
$\therefore$
$$v^2=2gl-2gl(1-\cos\theta)$$
or
$$v^2=2gl\cos\theta$$
or
$$\cos\theta=\frac{v^2}{2gl}. \qquad \ldots(4)$$

From (2), (3) and (4), we get
$$T=m\left[2g-2g+\frac{3v^2}{2l}\right]$$
$$=\frac{3mv^2}{2l}.$$

Ex. 3. *A heavy particle is free to move in a vertical circle of radius l, the particle is projected with velocity u from the lowest point A of the circle. If v be its velocity when it has risen a vertical distance h, prove that $v^2=u^2-2gh$. If it just reaches a point P, show that*
$$u=\sqrt{\frac{g}{l}}\,AP.$$

Let the vertical distance of P above A be h so that

$$AN=h$$

$\therefore$ $$v^2=u^2-2gh. \text{ (Art. 7·21)}$$

Again, let $v=0$ at P.

$\therefore$ $$u^2=2gh$$
$$=2g.AP\cos\theta,$$

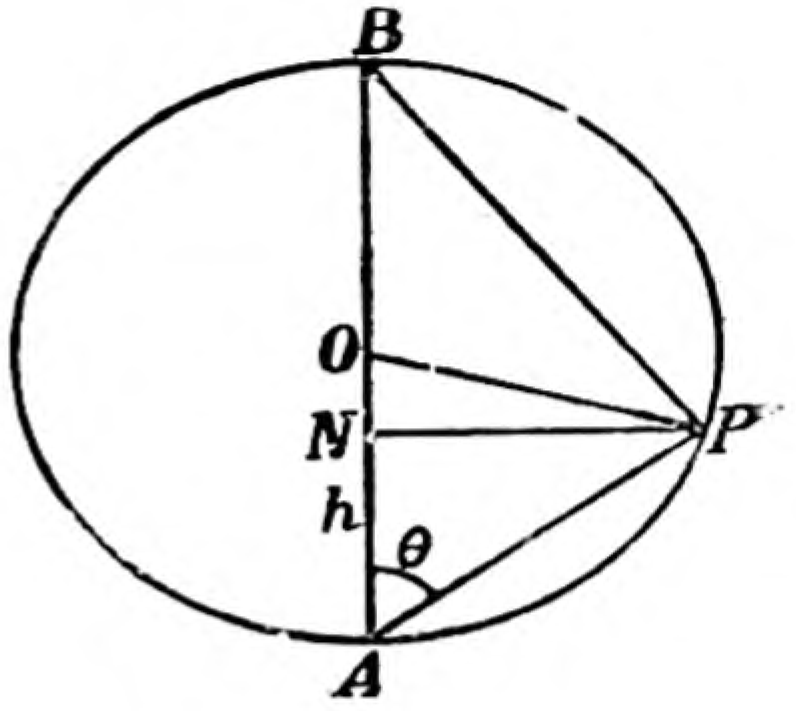
where $\angle PAN=\theta$

$$=2g.\ AP.\frac{AP}{AB}$$

$$=2g.\ \frac{AP^2}{2l}$$

$$=\frac{g}{l}.AP^2.$$

$\therefore$ $$u=\sqrt{\frac{g}{l}}.AP.$$



Exercise VII (B)

1. (*a*) A particle hangs by a string of length a from a fixed point O and is given an initial velocity $\sqrt{2gh}$. Prove that, if the particle makes complete revolutions, h is at least equal to $\frac{5a}{2}$.

(*J. & K.*)

(*b*) Show that the velocity with which a particle hanging from a fixed point by a string of length a must be started so as to describe a complete vertical circle must not be less than $\sqrt{5ag}$. (*J. & K.*)

2. The roadway of a bridge over a canal is in the form of a circular arc of radius 50 ft What is the greatest velocity with which a motor cycle can cross the bridge without leaving the ground at the highest point ?

3. A tumbler containing some water is swung in a vertical circle of radius r by means of a string attached to it. Show that in order to prevent the water from spilling outside, the velocity at the lowest point must be at least $\sqrt{5gr}$.

4. A heavy particle is attached by a string, 10 ft. long, to a fixed point and swung round in a vertical circle. Find the tension and velocity at the lowest point of the circle so that the particle may just complete the revolutions. (*P.U. B.Sc. 1961*)

5. A trick cyclist rides inside a sphere of radius 12 ft. with head downwards past the highest point. Find the least speed he must then have in order that he can perform the feat. Assume the mass of the machine and the rider to be concentrated at a point 4 ft. from the track.

6. A heavy particle hangs by a string from a fixed point and rotates in a vertical circle. Show that the sum of the tensions of the string when the particle is at the opposite ends of a diameter is the same for all diameters.

7. (*a*) A particle is projected along the inner side of a smooth vertical circle of radius a, the speed at the lowest point being u. Show that if

$$2ag < u^2 < 5ag,$$

the particle will leave the circle before arriving at the highest point and then describe a parabola of latus rectum

$$\frac{2(u^2-2ag)^3}{27a^2g^3}. \qquad \textit{(P.U.)}$$

(*b*) If $u^2=4gr$, determine the velocity of the particle on leaving the circular path.

Also find the angle that this velocity makes with the horizontal. ***(P.U. B.Sc., 1960)***

8. A particle is projected along the inner surface of a smooth vertical circle of radius a, its velocity at the lowest point being $\frac{1}{5}\sqrt{95ga}$; show that it will leave the circle at an angular distance $\cos^{-1}\frac{3}{5}$ from the highest point and its velocity then is $\frac{1}{5}\sqrt{15ga}$.

9. The bob of a freely hanging simple pendulum of length l is projected from its lowest position with speed $\sqrt{6gl}$, find the velocity of the bob and tension in the string when it makes an angle of 120° with the downward vertical. Weight of the bob is W. ***(P.U.)***

10. A heavy particle hanging from a fixed point by a light inextensible string of length a is projected horizontally with a velocity u. Show that during the circular motion the tension of the string at any time is proportional to the depth of the particle at that moment below a certain horizontal line, and find the values between which u must lie that the string may become slack. ***(P.U.)***

11. A stone of weight w is tied to one end of a string and is describing a vertical circle, the other end of the string being fixed. If the maximum speed of the stone is twice the minimum speed, prove that when the string is horizontal, its tension is

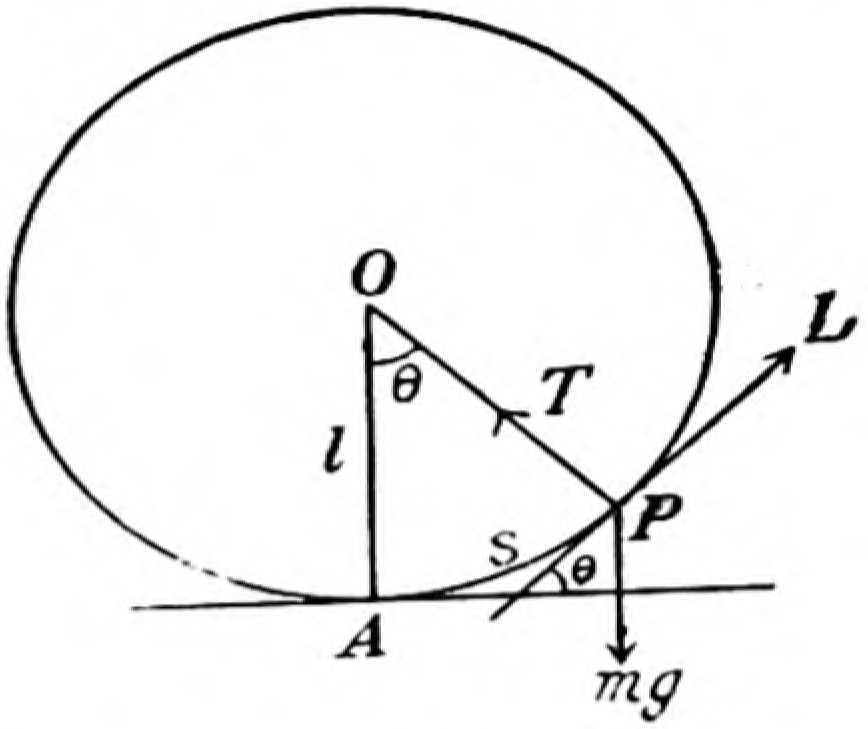
$$\frac{10w}{3}.$$

Also find the tension when it makes an angle of 45° with the downward vertical. ***(P.U. 1967)***

7·7 Period of a small oscillation of a Simple Pendulum.

A heavy particle is tied to one end of a light inextensible string the other end of which is fixed, and oscillates in a vertical circle, through a small angle. To find the period. (*Delhi, 1956 ; Panjab, 1953 ; Agra*)

Let O be the fixed point, P the position of the particle at time t, A the lowest point, arc AP$=s$ and $\angle$AOP$=\theta$.



The forces acting on the particle are its weight mg acting vertically downwards and the tension T of the string along PO.

The equation of motion along the tangent PL at P is

$$m\frac{d^2s}{dt^2}=-mg\sin\theta$$

Since θ is a small angle measured in radians, and

$$\underset{\theta\to 0}{\text{Lt}}\frac{\sin\theta}{\theta}=1,\ \sin\theta \text{ may be replaced by } \theta$$

$$\therefore\quad \frac{d^2s}{dt^2}=-g\theta.$$

Let l be the length of the string.

Then $$\theta=\frac{s}{l}$$

$$\therefore\quad \frac{d^2s}{dt^2}=-\frac{g}{l}s \qquad \left(\text{Form } \frac{d^2x}{dt^2}=-\mu x\right)$$

which shows that the motion is Simple Harmonic.

Time period $$=\frac{2\pi}{\sqrt{\frac{g}{l}}}$$

$$=2\pi\sqrt{\frac{l}{g}}.$$

The time of an oscillation is independent of amplitude (*i.e.* the motion of a simple pendulum is *isochronous*).

7·71 The Seconds' Pendulum

A seconds' pendulum is that which oscillates from rest to rest (*i.e.* makes half a complete oscillation) in one second.

$$\therefore\quad 1=\pi\sqrt{\frac{l}{g}},$$

where l is the length of the seconds pendulum.

$$\therefore\quad l=\frac{g}{\pi^2}=\frac{32{\cdot}2}{9{\cdot}87}\text{ ft.}$$

$$=39{\cdot}14 \text{ inches, in F.P.S. system.}$$

This gives the length of the correct seconds' pendulum.

Taking $g=981$ cm /sec^2.,
$l=99{\cdot}4$ cm. in C.G.S. system.

7·8 Loss or gain of Oscillations

We have seen that for a simple pendulum :

$$t=\pi\sqrt{\frac{l}{g}}.$$

where l is the length of the simple pendulum and t the time of one beat.

Let the pendulum make n beats in time T.

Then
$$\mathrm{T}=nt=n\pi\sqrt{\frac{l}{g}}$$

$$\therefore \qquad n=\frac{\mathrm{T}}{\pi}\sqrt{\frac{g}{l}} \qquad \text{...(I)}$$

so that n, the number of beats (or frequency), depends upon the length l and the value of g at a place. Let dn denote the increase in the number of beats corresponding to an increase dg in g and dl in l.

Taking logarithms of (I) :

$$\log n=\log \mathrm{T}-\log \pi+\tfrac{1}{2}\log g-\tfrac{1}{2}\log l.$$

Taking differentials, we get

$$\frac{1}{n}\,dn=\frac{1}{2}\left(\frac{1}{g}dg-\frac{1}{l}dl\right), \qquad \text{...(II)}$$

since T and π are constants.

Case I. If only the length l varies, $dg=0$.

$$\therefore \qquad \frac{dn}{n}=-\frac{1}{2}\,\frac{dl}{l}$$

$$\therefore \qquad dn=-\frac{n}{2}\,\frac{dl}{l}. \qquad \text{...(III)}$$

The negative sign shows that there is a decrease in the number of beats in any interval of time corresponding to an increase in the length of the pendulum. Thus the clock becomes slow.

Case II. If only g varies, $dl=0$.

$$\therefore \qquad \frac{dn}{g}=\frac{1}{2}\,\frac{dg}{g}$$

$$\therefore \qquad dn=\frac{n}{2}\,\frac{dg}{g}. \qquad \text{...(IV)}$$

Thus an increase in the value of g makes the clock go fast.

7·9 Change in g due to position

(*i*) *For points outside the surface of the earth g varies inversely as the square of the distance from the centre of the earth* ; therefore if r is the distance from the centre of earth of a point above the earth's surface,

$$g=\frac{\mu}{r^2}, \text{ when } \mu \text{ is a constant.}$$

Taking logarithms,

$$\log g = \log \mu - 2 \log r$$

$$\therefore \quad \frac{dg}{g} = -\frac{2dr}{r}.$$

$\therefore$ If h is the height above the earth's surface, then

$$dr = h.$$

$$\therefore \quad \frac{dg}{g} = -\frac{2h}{r}$$

[From (IV) of Art. 7·8]

$$\therefore \quad \frac{dn}{n} = \frac{1}{2}\frac{dg}{g} = -\frac{h}{r}$$

$$\therefore \quad dn = -\frac{nh}{r}.$$

Hence when the clock is taken to a high mountain, the number of oscillations is lost so that the clock becomes slow. *(Delhi, 1962)*

7·91 *(ii) For points inside the surface of the earth, g varies directly as the distance from the centre of the earth.*

$$\therefore \quad g = \mu r$$

$$\therefore \quad dg = \mu dr = \frac{g}{r} dr$$

$$\therefore \quad \frac{dg}{g} = \frac{dr}{r}.$$

If D be the depth below the surface, $dr = -\text{D}$.

$$\therefore \quad \frac{dg}{g} = -\frac{\text{D}}{r}$$

[From (IV) of Art. 7·8]

$$\therefore \quad \frac{dn}{n} = \frac{1}{2}\frac{dg}{g} = -\frac{\text{D}}{2r}.$$

$$\therefore \quad dn = -\frac{n\text{D}}{2r}. \qquad \text{...(VI)}$$

Hence even when the clock is taken to a deep mine, the number of oscillations is lost so that the clock again becomes slow.

(Delhi, 1962 ; Panjab, 1959 S)

Ex. 4. *Find how many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900 : 901.*

(P.U., J. & K.)

By Art. 7,

$$dn = -\frac{n}{2} \cdot \frac{dl}{l}. \qquad \text{...(I)}$$

Here n = No. of seconds in a day = 86400.

$$l : l + dl = 900 : 901$$

whence

$$\frac{dl}{l} = \frac{1}{900}.$$

$$\therefore \qquad dn = -\frac{86400}{2} \times \frac{1}{900} = -48.$$

Hence the clock loses 48 secs. per day.

Ex. 5. *A clock keeps correct time at a place where $g=32{\cdot}1$ ft./sec^2., how many seconds would it gain or lose per day if taken to a place where $g=32{\cdot}2$ ft./sec^2 ?*

Here $$g=32{\cdot}1 \text{ ft./sec}^2.$$

and $$g+dg=32{\cdot}2 \text{ ft./sec}^2.$$

$$\therefore \qquad \frac{g+dg}{g}=\frac{32{\cdot}2}{32{\cdot}1}$$

or $$\frac{dg}{g}=\frac{32{\cdot}2}{32{\cdot}1}-1$$

$$=\frac{0{\cdot}1}{32{\cdot}1}$$

$$=\frac{1}{321}.$$

Now $$\frac{dn}{n}=\frac{1}{2}\;\frac{dg}{g}.$$

$$\therefore \qquad dn=\frac{n}{2}\;\frac{dg}{g},$$

where $$n=\text{No. of seconds in a day}$$

$$=24\times 60\times 60$$

$$=86400$$

$$\therefore \qquad dn=\frac{86400}{2\times 321}$$

$$=134\;6$$

which is positive.

Hence the clock gains 134·6 secs. per day.

Ex. 6. *A seconds pendulum loses 10 secs. a day at a place where g is 32 ft./sec^2. What change is necessary to make it accurate ?*

Let l be the correct length and $l+dl$ the false length of the pendulum.

Then $$l=\frac{g}{\pi^2}$$

and $$dn=-\;\frac{n}{2}\cdot\frac{dl}{l}, \text{ where } n=86400$$
$$\text{and } dn=-10$$

$$\therefore \qquad -10=-\;\frac{86400}{2}\cdot\frac{dl}{g/\pi^2}$$

$$\therefore \qquad dl=\frac{20g}{86400\pi^2}=\frac{20\times 32}{86400\times(3{\cdot}1416)^2}\text{ft.}$$
$$=0{\cdot}009 \text{ inch.}$$

Ex. 7. *If a pendulum beating seconds at the foot of a mountain loses 8 seconds in a day when taken to its summit, find approximately the height of the summit, assuming the radius of the earth to be 4,000 miles, and neglecting the attraction of the mountain.*

Here
$$n=24\times60\times60$$
$$dn=-8$$
$$r=4000\times1760\times3 \text{ ft.}$$

Since
$$dn=-\frac{n}{r}h$$
$$\therefore \qquad -8=-\frac{24\times60\times60}{4000\times1760\times3}h$$
$$\therefore \qquad h=1955\tfrac{5}{9}\text{ ft.}$$

Ex. 8. *A seconds' pendulum was too long on a given day by a quantity* α *; it was then over-corrected so as to be too short during the next day ; prove that* l *being the correct length, the number of minutes gained in the two days was* $\frac{1080\alpha^2}{l^2}$ *nearly.* (*P.U. 1960*)

$$\pi\sqrt{\frac{l}{g}}=1. \qquad \ldots(\text{I})$$

Let t be the time of one swing when l becomes $l+\alpha$

Then
$$t=\pi\sqrt{\frac{l+\alpha}{g}}.$$

If n be the number of oscillations in a day,
$$nt=86400$$
$$\therefore \qquad n=\frac{86400}{t}$$
$$=\frac{86400}{\pi}\sqrt{\frac{g}{l+\alpha}}$$
$$=\frac{86400}{\pi}\sqrt{\frac{g}{l}}\cdot\sqrt{\frac{l}{l+\alpha}}$$
$$=86400\sqrt{\frac{l}{l+\alpha}}, \qquad \text{from (I)}$$
$$=86400\left(1+\frac{\alpha}{l}\right)^{-\frac{1}{2}}$$
$$=86400\left(1-\frac{\alpha}{2l}+\frac{3\alpha^2}{8l^2}+\ldots\ldots\right)$$

$$\therefore \qquad \text{loss in a day}=86400\left(\frac{\alpha}{2l}-\frac{3\alpha^2}{8l^2}\right) \text{ vibrations,}$$

Similarly when l becomes $l-\alpha$, loss in a day

$$=86400\left(-\frac{\alpha}{2l}-\frac{3\alpha^2}{8l^2}\right),$$

i.e., gain in a day$=86400\left(\frac{\alpha}{2l}+\frac{3\alpha^2}{8l^2}\right)$ vibrations.

$\therefore$ gain in the number of vibrations in two days

$$=\frac{3\alpha^2}{8l^2}\times 86400$$

Now gain in one vibration means one second.

$$\therefore \qquad \text{time gained}=86400\times\frac{3\alpha^2}{4l^2} \text{ seconds}$$

$$=1080\frac{\alpha^2}{l^2} \text{ minutes.}$$

Ex. 9. *A clock, which at the surface of the earth gains 10 sec. a day, loses 10 sec. a day when taken down a mine. Compare the accelerations due to gravity at the top and bottom of the mine and find its depth. Take the radius of the earth to be 4,000 miles.* **(*P.U.*)**

Since it is gaining at the surface ; it means the value of gravity is above normal, say $g+dg$ **...(I)**

$$\text{Also, } \because \qquad dn=\frac{n}{2}\cdot\frac{dg}{g}$$

$$\therefore \qquad 10=\frac{86400}{2}\cdot\frac{dg}{g} \quad \text{or} \quad \frac{dg}{g}=\frac{1}{4320}$$

$$\text{or} \qquad \frac{g+dg}{g}=\frac{4321}{4320} \qquad \textbf{...(II)}$$

Also, in a mine let gravity be $g-\delta g$, where the loss being 10 seconds,

$$\frac{dn}{n}=-\frac{\delta g}{2g} \quad \text{or} \quad \frac{\delta g}{g}=\frac{1}{4320}$$

$$\therefore \qquad \frac{g-\delta g}{g}=\frac{4319}{4320} \qquad \textbf{...(III)}$$

Again, for a mine of depth D, $dn=-\frac{nD}{2r}$

$$\text{giving,} \qquad -10=-\frac{86400}{2}\times\frac{D}{4000}$$

$$\text{or} \qquad D=\frac{400}{432}=\frac{25}{27} \text{ mile}=4888\tfrac{8}{9} \text{ ft.}$$

Also the ratio of the values of gravities

$$=\frac{g+dg}{g-\delta g}=\frac{4321}{4319}.$$

Exercise VII (C)

1. A clock with a seconds pendulum loses 20 seconds per day at the place where acceleration due to gravity is 32 ft./sec^2. Find what change (*i*) in length, (*ii*) in gravity is necessary to make it accurate.

2. A clock with a seconds pendulum is gaining two minutes a day. Prove that the length of the pendulum must be increased by 0·28 cm. to make it go correctly. (*P.U.*)

3. Show that an incorrect seconds pendulum of a clock which loses x seconds a day must be shortened by $\frac{x}{432}$ per cent of its length in order to keep correct time.

4. A pendulum whose length is l makes m oscillations in 24 hours. When its length is slightly altered it makes $m+n$ oscillations in 24 hours. Show that the diminution of length is $\frac{2nl}{m}$ nearly. (*P.U.*)

5. Show that the height of a hill at the summit of which a seconds pendulum loses n beats in 24 hours is approximately $245n$ feet.

6 Assuming that acceleration due to gravity varies inversely as the square of the distance from the centre of the earth, find approximately the height of a mountain at the top of which a pendulum, which beats seconds at sea level, loses 20 seconds a day. The radius of the earth may be taken as 4,000 miles.

7. A pendulum beats seconds accurately at a place where the acceleration due to gravity is 32 ft./sec^2. If taken to a place where the value of this acceleration is 32·2 ft./sec^2., will it gain or lose, and how many seconds in 24 hours ?

8. A pendulum is carried to the top of a mountain half a mile high, how many seconds will it lose per day ? By how much must its present length be shortened so that it may beat seconds at the top of the mountain ?

9. A simple pendulum swings under gravity in such a manner that when the string is vertical, the force which it exerts on the bob is twice its weight ; prove that the greatest inclination of the string to the vertical is 60°.

10. A seconds pendulum hangs against a wall inclined at an angle θ to the horizontal. Show that the time of a complete small oscillation is

$$2\pi\sqrt{\frac{l}{g\sin\theta}}.$$

11. A pendulum of length l has one end of the string fastened to a peg on a smooth plane inclined to the horizontal at an angle α. With the string and the weight on the plane, its time of oscillation is t seconds. If a pendulum of length l' oscillates in one second when suspended vertically, prove that

$$\alpha=\sin^{-1}\left(\frac{l}{l'l^2}\right).$$

(*P.U.*)

Miscellaneous Exercise V

1. A particle moving on the inside of a smooth hollow sphere of radius r, describes a horizontal circle at a depth of $\frac{1}{2}r$ below the centre of the sphere. Show that its speed is $\frac{1}{2}\sqrt{6gr}$.

2. A body moving in a horizontal circle makes n revolutions per second on the inner surface of a smooth hollow cone of vertical angle 2α with axis vertical and vertex downwards. Show that the distance of the body from the axis of the cone is

$$\frac{g \cot \alpha}{4\pi^2 n^2}.$$

3. Two masses m_1 and m_2 are connected by a string of length l passing through a small fixed smooth ring. Find the least velocity of revolution of m_1 in a conical pendulum, in order that m_2 may hang at rest at a depth x below the ring.

4. A cyclist's circular racing track is the part of a cone with axis vertical, vertex downwards, semi-vertical angle 60° and radius 120 ft. If the coefficient of friction be 0·3 ; find the maximum and minimum speeds possible on the track. (*Public Service Commission*)

5. A car takes a banked corner of a racing track at speed V, the lateral gradient α being designed to reduce the tendency to side slip to zero for a lower speed v. Show that the coefficient of friction to prevent side slip for the greater speed V must be at least

$$\frac{(V^2-v^2)\sin\alpha\cos\alpha}{V^2\sin^2\alpha+v^2\cos^2\alpha}$$

(*P.U., 1961 S*)

6. If a simple pendulum oscillates through an angle α on either side of the mean position, show that the period of an oscillation is

$$2\pi\sqrt{\frac{l}{g}\left(1+\frac{1}{4}\sin^2\frac{\alpha}{2}\right)},$$

where α is so small that $\sin^4\frac{\alpha}{2}$ and higher powers of $\sin\frac{\alpha}{2}$ can be neglected. (*P.U., 1961 S*)

7 In an oscillatory pendulum the tension of the string when the bob is in its lowest position is n times the weight of the bob. Prove that the angle of swing on each side of the vertical is

$$\cos^{-1}\left(\frac{3-n}{2}\right).$$

(*P.U.*)

8. A particle attached to a fixed point O by an inelastic string of length r is let fall from a point in the horizontal through O at a distance V cos θ from O. Show that the velocity of the particle when it is vertically below O is $\sqrt{2gr\,(1-\sin^2\theta)}$. Find also the tension there. (*P.U. 1957 S*)

9. A particle attached to a fixed point O by a string of length l is held up with the string stretched and horizontal and then let go. Prove that when the string makes angle θ with the horizontal the resultant acceleration is

$$g\sqrt{1+3\sin^2\theta}.$$ (*P.U., 1968*)

10. A heavy particle is free to move in a vertical circle of radius l and is projected with a velocity u from the lowest point A of the circle. If it just reaches a point B, show that

$$u=\sqrt{\frac{g}{l}}\ \text{AB}.$$

(*i*) Tension at A is to the tension at B as

$$\frac{2l^2+2\text{AB}^2}{2l^2-\text{AB}^2}.$$

11. A particle slides outside a fixed smooth circle in a vertical plane starting at rest at the top and another is projected from the lowest point inside the circle with just sufficient velocity to carry it to the top. Prove that both leave the circle at the same point and proceed to describe parts of the same parabola. (*P.U., 1961*)

12. A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which impart to it a velocity $\sqrt{2gl}$. Prove the chord becomes slack when the particle has risen to a height $\frac{2l}{3}$ above the fixed point and find the highest point of the subsequent path.

13. A simple pendulum is started to make complete revolutions in a vertical plane. What must be the least velocity of projection ?

In the subsequent motion ω_1 and ω_2 are the greatest and least angular velocities and T_1 and T_2 are the greatest and least tensions. Prove that when the pendulum makes an angle θ with the vertical, the angular velocity is

$$\sqrt{\omega_1{}^2\cos^2\frac{\theta}{2}+\omega_2{}^2\ \sin^2\frac{\theta}{2}}.$$

and that tension is $T_1\cos^2\frac{\theta}{2}+T_2\sin^2\frac{\theta}{2}$. (*P.U.*)

14. A heavy particle of mass m makes complete revolutions in a smooth circular tube fixed in a vertiral plane. Its greatest speed is n times its least speed. Prove that the pressure in the tube when the particle is moving vertically is

$$\frac{2(n^2+1)mg}{n^2-1},$$

that its speed then is

$$\sqrt{\frac{n^2+1}{2}}$$ times its speed at the highest point. (*P.U.*)

15. A particle which hangs from a point by a string of length a is projected horizontally from the position of equilibrium with a velocity due to a height $(a+b)$. If $2b < 3a$ show that the string will be loose for a time t given by the equation

$$27ga^2t^2=8b\,(9a^2-4b^2).$$

16. A bead is free to slide on a smooth circular wire and is projected from the lowest point with a velocity just sufficient to carry it to the highest point. Show that the reaction changes sign after a time

$$\sqrt{\frac{a}{g}}\ (\log\,(\sqrt{5}+6).$$

17. A thin smooth tube forms an arc of a circle of radius a in a vertical plane, its ends A and B being open and at the same level and the arc AB subtends an angle $(2\pi-2\alpha)$ at the centre. If a particle is started at the bottom of a tube with a velocity V, prove that in order that it may re-enter the tube B after leaving the tube at A

$$V^2=ag\left(\sec\,\alpha+4\cos^2\frac{\alpha}{2}\right).$$

18. A hollow cylinder of internal radius a rotates round its horizontal axis with constant angular velocity ω. Show that a small particle inside the cylinder will start to slide when its radius makes an angle θ with the downward vertical.

ANSWERS

Exercise VII (A), Page 155

1. (*a*) $\cos^{-1}\frac{2}{3}$.

(*b*) (*i*) $\cos^{-1}\frac{u^2+2gr}{3gr}$; (*ii*) $\cos^{-1}\left(\frac{2}{3}\cos\alpha\right)$.

(*c*) (*i*) L.R. $=\frac{16}{27}$; (*ii*) $\frac{2(u^2+2gr)^3}{23g^3r^2}$; (*iii*) $\frac{16r\cos^3\alpha}{27}$.

Exercise VII (B), Page 161

2. 40 f ps. **4.** $T=6\,mg,\ u=40$ f.p.s. **5.** 16 f.p.s.

7. $v=\sqrt{\frac{2}{3}gr},\ \theta=\cos^{-1}\frac{2}{3}$. **9.** $T=\frac{5}{2}w,\ v=\sqrt{3gl}$

10. $2ga < u^2 < 5ga$. **11.** $\left(\frac{10}{3}+\frac{3}{\sqrt{2}}\right)w$.

Exercise VII (C), Page 169

1. (*i*) ·018 in. (*ii*) g must be increased by 0·015.

6. $\frac{25}{27}$ mile or $4888\frac{8}{9}$ feet. **7.** gains 270 secs.

8. 10·8 secs ; $\frac{l}{4000}$.

Miscellaneous Exercise VI, Page 170

3. $\frac{1}{2\pi}\sqrt{\frac{m_2g}{m_1(l-x)}}$. **4.** 64 ft. and 30 ft.

8. $T=mg\sqrt{3-2\sin^2\theta}$. **12.** $\frac{9l}{16}$.

CHAPTER VIII

WORK, POWER, ENERGY AND IMPULSE

(A) Work and Power

8·1 Mechanical Work. A force is said to do *work* when a body moves under its action.

The work done by a force is called "Mechanical Work" as distinguished from other types of work.

Def. *The work done by a constant force is defined as the product of the force and the displacement of its point of application in its direction.*

Thus, if the displacement S takes place in a direction making $\angle\theta$ with the force F ; then

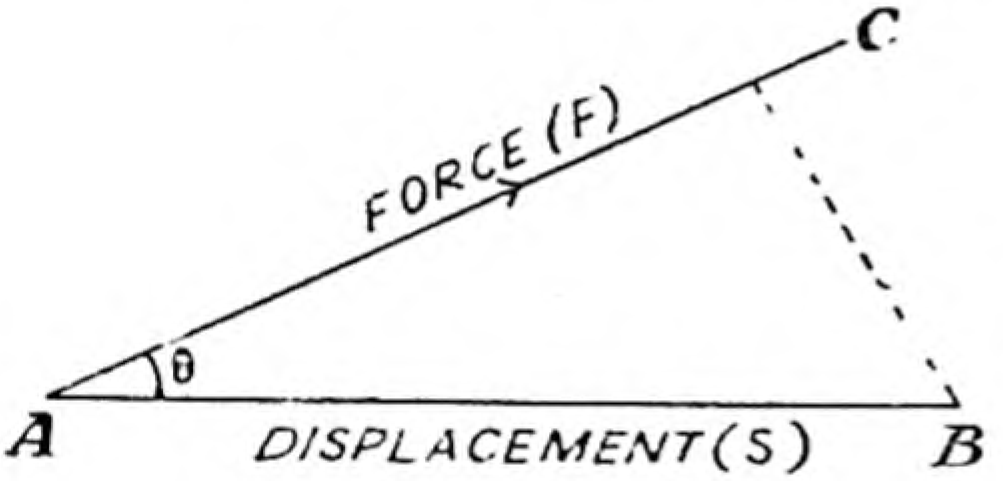
$$\text{Work} = F \times \text{resolved part of S along AC.}$$

$$\therefore\ W = F\,.\,S\cos\theta$$

or

$$S\,.\,F\cos\theta. \qquad \ldots(I)$$

Thus *work done is also equal to the product of the "displacement and the resolved part of force in its direction."*



8·11 Simple Deductions. 1. If $\theta=0$, *i.e.*, the displacement takes place in the direction of the force, $W=F\,.\,S=\text{Force}\times$ Displacement ; ...(II)

2. If $\theta=180^\circ$, *i.e.*, displacement in the direction opposite to that of force, $W=-F\,.\,S$; ...(III)

3. If $\theta=90^\circ$, *i.e.*, the displacement is at right angles to the force, $W=F\cos 90^\circ=0$; ...(IV)

so that the force does not work in this case.

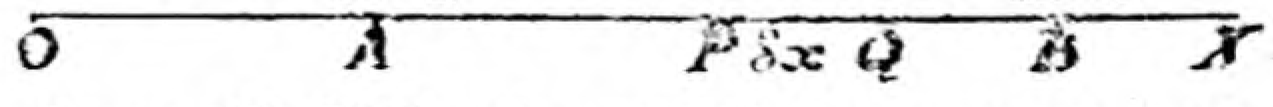
Ex. A body falls from rest under gravity from a height h. The amount of work done by its weight is mgh. If a body is lifted through a height h ; then since its direction of motion is opposite to that of the force of weight, the work mgh is done against the weight or we may say work done by the weight is $-mgh$.

8·12 Work done by a Variable Force.

Let a variable force act along the line OX and suppose that the particle also moves along OX. If F is the magnitude of the force at the point P at a distance a from O, δW, the small amount of work done by the force, considered constant for an indefinitely small displacement δx is $F\,.\,\delta x$

or

$$\delta W = F\,.\,\delta x.$$



Hence the total work done as the particle moves from $x=a$ to $x=b$

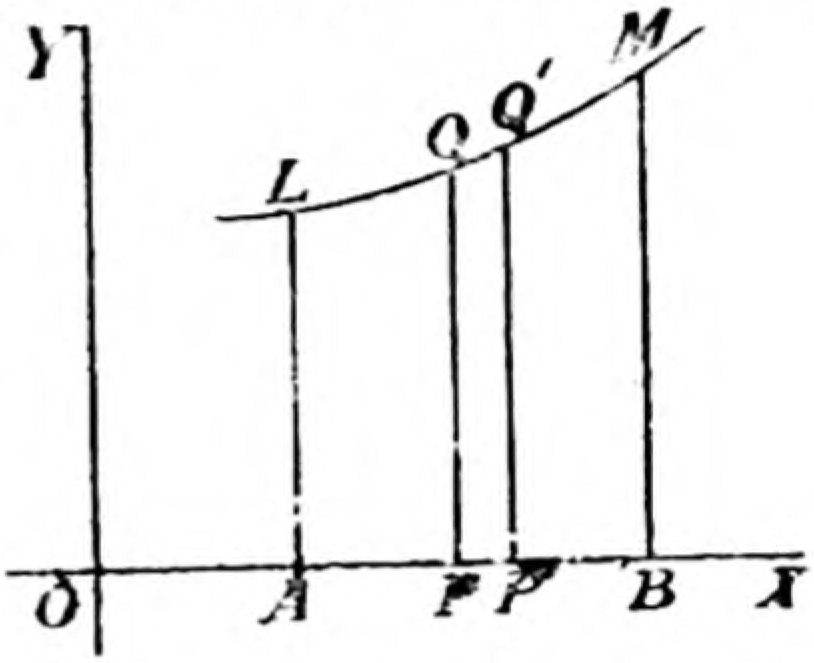
is $$W=\int_a Fdx.$$

8·13 Graphical Representation of Work

When the force F is variable, it may be taken as dependent of x (the distance) so that

$$F=f(x).$$

If we draw a graph by plotting the distance x moved by the point of application of the force along an axis OX and the values of the force for different values of x parallel to a particular axis OY. The curve LQM obtained in this way is called **Force-distance Graph**.



The area of the strip PQ′ of breadth dx is PQ . dx or F . dx, where F is the value of the force when $x=$OP.

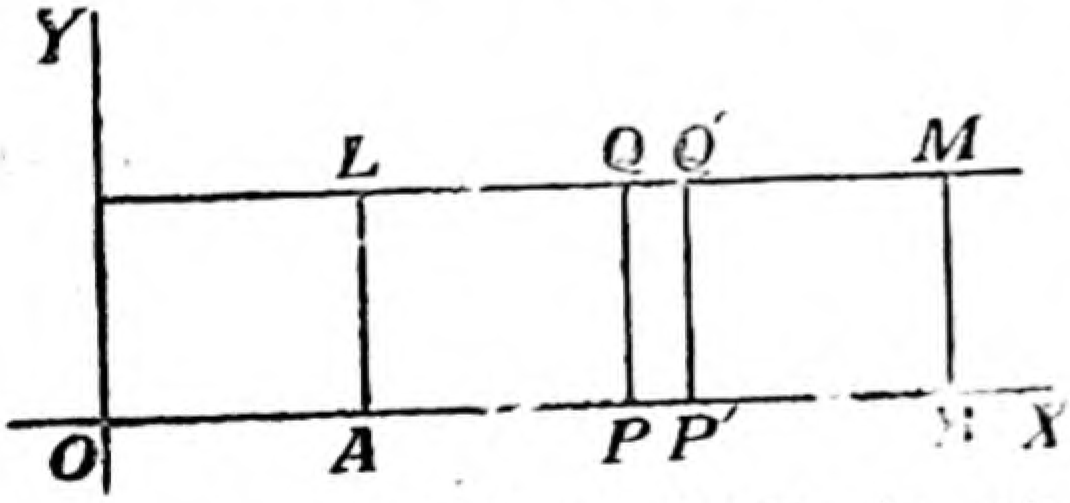
If $OA=x_1$, $OB=x_2$

the work done by F between these limits

$$=\int_{x_1}^{x_2} Fdx$$

which represents the area under the curve LQM.

Hence the area under the force-distance graph gives the work done by the force.



In particular when the force is *constant* the force-distance graph is a straight line parallel to the axis of x and the work done is represented by the area of a rectangle and is equal to

$$F(x_2=x_1).$$

8·14 Units of Work.

(*i*) The absolute unit of work in the F.P.S. system is the work done by a force of one poundal in moving its point of application through 1 foot in the direction of the force.

This unit of work is called a **Foot Poundal**.

(*ii*) The absolute unit of work in the C.G.S. system is the work done by a force of 1 dyne in moving its point of application through 1 centimetre in the direction of the force.

This unit is called an **Erg**.

(*iii*) The unit of work used by engineers is called a Foot-Pound. It is the work done in raising a mass of 1 lb. vertically through 1 foot.

Since 1 lb. wt. $=g$ poundals,

1 ft. lb. $=g$ ft. poundals.

The C.G.S. gravitation unit is called **Kilogram-metre.** This is the work done in raising a kilogram (1000 grammes) through one metre (100 cm.).

$\therefore$ A kgm. metre $=10^5$ gramme centimetres.

One gramme centimetre $=g(981)$ ergs.

8·2 Work done in stretching an elastic string.

Let l be the natural length of the string whose one end is fixed. Let it be stretched to a length $l+x$. The tension T of the string then is

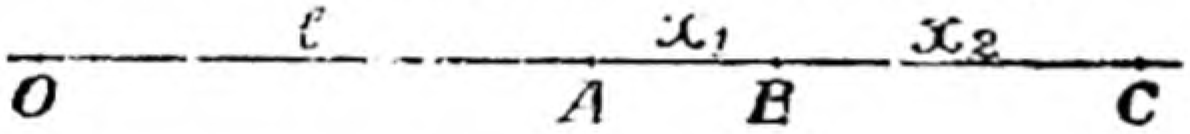
$$T=\frac{\lambda x}{l}.$$

As the string is stretched the force applied at any instant is always equal to the tension of the string.

Therefore δW the small amount of work done in stretching the string a further distance δx is

$$T\delta x \qquad \text{or} \qquad \delta W=T\delta x.$$

The total work done in stretching the string from the natural length l to a length $l+x$ is given by



$$W=\int_0^x T\,dx$$

$$=\int_0^x \frac{\lambda x}{l}\,dx$$

$$\therefore \quad W=\frac{\lambda}{2}\,\frac{x^2}{2}. \qquad \text{...(I)}$$

(*ii*) The work done in increasing the extension from x_1 to x_2 is given by

$$W=\int_{x_1}^{x_2} \frac{\lambda}{l}\,x\,dx=\frac{\lambda}{l}\left|\frac{x^2}{2}\right|_{x_1}^{x_2}$$

$$=\frac{\lambda}{l}\cdot\frac{(x_2{}^2-x_1{}^2)}{2} \qquad \text{...(II)}$$

$$=\frac{\lambda}{l}\left(\frac{x_2+x_1}{2}\right)\left(x_2-x_1\right). \qquad \text{...(III)}$$

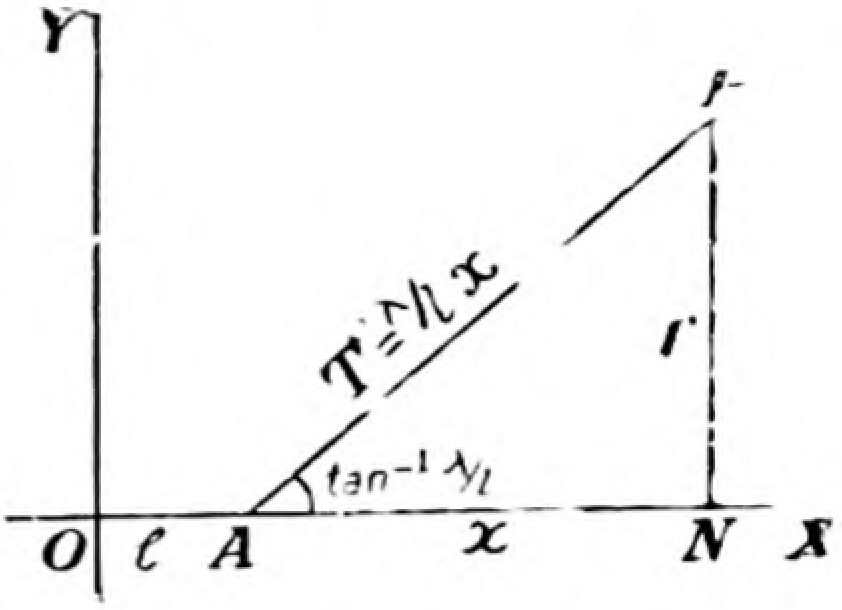
Now $\frac{\lambda x_2}{l}$ is the final tension (at C) and $\frac{\lambda x_1}{l}$ is the initial tension (at B).

$\therefore$ $\frac{\lambda(x_1+x_2)}{2}$ is the mean of the initial and final tensions.

Also (x_2-x_1) is the extension (BC).

Hence the work done is the product of the mean of initial and final tensions and the extension.

8·21 Graphical Method. Since $T=\frac{\lambda}{l}x$. The Force Distance graph is a straight line whose slope is $\frac{\lambda}{l}$.



Let OA$=l$ be the initial length and ON$=l+x$, the final length. The work done in stretching the string from a length l to a length $l+x$

$$=\text{area APN}=\tfrac{1}{2}\text{AN.NP}.$$

$$\therefore \quad W=\frac{1}{2}\;\frac{\lambda}{2}\;AN^2=\frac{1}{2}\;\frac{\lambda}{2}x^2.$$

Also $W=\frac{1}{2}.AN.NP=\frac{1}{2}x.T=\frac{1}{2}$(Tension × Extension).

(P U., 1952 S)

Ex. 1. *An elastic string of natural length 2 feet is stretched 1 inch by a weight of 1 lb. Find the work done in stretching it from a length of 2½ ft. to 3 ft.*

The fact that 1 lb. wt. stretches the string 1 inch enables us to find λ for

$$1.g=\frac{\lambda}{2}\;\frac{1}{12} \qquad \therefore \; \lambda=24g.$$

Initial tension for an extension of 6 inches is

$$T_1=\lambda\,\frac{\frac{1}{2}}{2}=6g.$$

Final tension for an extension of 1 feet is

$$T_2=\frac{\lambda}{2}.1=12g$$

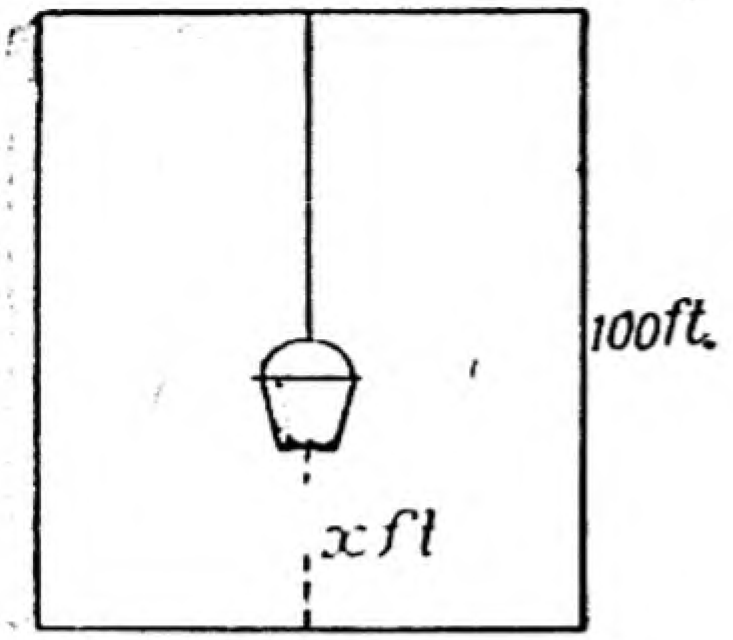
∴ the mean of the initial and final tension

$$=\frac{T_1+T_2}{2}=9g.$$

The extension$=$6 inches$=\frac{1}{2}$ ft.

∴ work done$=9\times\frac{1}{2}=4\frac{1}{2}$ ft. lb.

Ex. 2. *A man is drawing water from a well with a light bucket which leaks uniformly ; the pail when full weighs 50 lbs. and when it arrives at the top half of the water remains. Find the work done if the depth of well is 100 ft.* *(E.A.C.)*



Initial mass $=50$ lbs.
Final mass $=25$ lbs.

Leakage in travelling 100 ft. $=25$ lbs,

,, ,, ,, x ft. $=\frac{x}{4}$ lbs.

$\therefore$ Mass of the water at height x

$$=\left(50-\frac{x}{4}\right) \text{ lbs.}$$

$\therefore$ Force of weight at height x

$$x=\left(50-\frac{x}{4}\right) \text{ lbs. wt.}$$

which is a variable force.

$$\therefore \text{ Work done}=\int_0^{100}\left(50-\frac{x}{4}\right)dx=\left[50x-\frac{x^2}{8}\right]^{100}$$
$$=3750 \text{ ft. lbs.}$$

Exercise VIII (A)

1. Define work and find an expression for the work done by a variable force which moves a body in its own line of action.

Prove that the work done in stretching an elastic string of natural length d and modulus of elasticity λ from tension T_1 to tension T_2 is $\frac{d}{2\lambda}(T_2^2-T_1^2)$. *(P.U.)*

2. An elastic string of natural length 3 ft. is stretched 2 inches by a weight of 1 lb. hanging on it. Find the work done in stretching it from a length of $3\frac{1}{2}$ ft. to $4\frac{1}{2}$ ft. *(P.U.)*

3. A particle of mass m, moves from a distance x_1 to a distance x_2 towards the centre of the earth. The force of attraction varying inversely as the square of the distance from earth's centre, show that the work done is $mga^2\left(\frac{1}{x_2}-\frac{1}{x_1}\right)$ where a is the radius of the earth.

4. A man of weight w hangs at the end of a light elastic rope whose modulus is nw, the other end being fastened to a fixed point. He proceeds to climb up the rope. Prove that when he reaches the fixed point, he has done $\frac{2n+1}{2n+2}$ times the work he would have to do in climbing the same distance upon inextensible rope. *(P.U., 1961)*

5. A locomotive weighing 55 tons is subject to frictional and air resistance at the rate of 12 lbs. wt. per ton. What work must be done in moving it through half a mile with an acceleration of $\frac{3}{11}$

ft./sec.^{2} ? What amount of work per second will the force be doing at the instant when it has moved half a mile ?

6. A cork, 3 inches long is drawn out slowly from the cylindrical neck of a bottle. If the force exerted at any instant be proportional to the surface area of the cork in contact with the bottle, calculate the work done in drawing out the cork, assuming that initially the whole cork is in the bottle and that the pull at the start is 48 lbs., the effect of air pressure being neglected. (*P.U.*)

8·3 Power. Power is the rate of doing work, *i.e.* the amount of work done in unit time. The F.P.S. unit of power is the Horse Power which is equivalent to 550 ft. lbs. per second or 33000 ft. lbs. per minute.

The C.G.S. unit of power is the watt which is 10^7 ergs per second of 1 Joule per second.

One horse power is equivalent to 746 watts.

If a force of F lb. wt. is moving its point of application in the direction of the force with uniform speed v ft per second the work done per second or (the rate of work) is $F \times v$ ft. lb. and the H.P. is $\frac{Fv}{550}$.

If v, the speed of a train is uniform, then the pull of the engine is equal to the resistance R due to friction etc. and the H.P. is equal to $\frac{Rv}{550}$.

Ex. 3. *What is the Horse Power of an engine which can just keep a train of mass 150 tons moving at a uniform rate of 60 miles per hour, the resistance to the motion due to friction and air etc. being taken at 10 lbs. wt. per ton.*

Since the speed is uniform the pull of the engine is equal to the total resistance, *i.e.* 150×10 or 1500 lb. wt.

Now 60 miles per hour $= 88$ ft./sec.

$\therefore$ the work done per second $= 1500 \times 88$ ft. lb.

$$\therefore \quad \text{H.P.} = \frac{1500 \times 88}{550} = 240.$$

Ex. 4. *A train of mass 100 tons is ascending uniformly an incline of 1 in 280 and the resistance due to friction etc., is equal to 16 lbs. wt. per ton if the engine be of 200 H.P. and be working at full power, find the rate at which the train is going.*

R = resistance due to friction = 1600 lb. wt. ...(I)

G = component of the weight down the slope

$= 100 \times \frac{1}{280}$ tons $= 800$ lbs. wt. ...(II)

$\because$ the speed is constant, the pull of the engine must be equal to the sum of R and G.

$\therefore$ the pull $=1600+800=2400$ lb. wt.

But the total work which the engine can do is 200×550 ft. lbs. per sec.

$$\therefore \quad 2400v=200\times550$$

or

$$v=\tfrac{1100}{24} \text{ ft. per sec.}$$
$$=31\tfrac{1}{4} \text{ miles per hour.}$$

Ex. 5. *Find the least H.P. of an engine which is capable of generating in a train of mass 100 tons, a velocity of 30 miles per hour in 4 minutes on a level line, the resistance due to friction etc., being equal to 8 lbs. wt. per ton and the pull of the engine being assumed constant.*

30 miles per hour $=44$ ft. per sec.

Since a velocity of 44 ft. per second is generated is 240 seconds, the acceleration of the train $=\frac{11}{60}$. $(\because\ 44=0+f.240)$.

Let the force exerted by the engine be P poundals, the resistance due to friction is 800 lbs. wt. Hence the net force on the train is $P-800g$ poundals.

$$\therefore \quad P-800g=100\times2240\times\tfrac{11}{60}$$
$$\therefore \quad P=800(g+\tfrac{154}{3}) \text{ poundals}$$
$$=800\times\tfrac{125}{48} \text{ lbs. wt.}$$

$\therefore$ the work done per second when the train is moving at 30 m.p.h.

$$=800\times\tfrac{125}{48}\times44 \text{ ft. lbs.}$$

$$\therefore \quad \text{H.P.}=\frac{800\times125\times44}{48\times550}=166\tfrac{2}{3}.$$

Exercise VIII (B)

1. The total mass of an engine and train is 200 tons. What is the H.P. of the engine if it can just keep the train moving at a uniform speed of 60 m.p.h. on the level, the resistances due to friction etc., amounting to 10 lbs. wt. per ton ?

2. A steam crane of 6 horse-power raises a load to a height of 100 ft. in 5 minutes. What is the greatest possible weight of the load ?

3. Find the uniform force that will move 1 lb. mass from rest through 1 foot in 1 second.

If this force is exerted while the mass moves through 75 yards from rest ; find the number of foot poundals of work done by the force and the maximum horse-power attained. *(P.U. B.Sc , 1960)*

4. An engine working at 500 H.P. pulls a train of 200 tons along a level track, the resistance being 16 lbs. per ton. When the velocity of the train is 30 m.p.h., find its acceleration. *(P.U.)*

5. A train of mass 200 tons is drawn up an incline of 3 in 500, at the rate of 30 m.p.h., the engine exerting 600 H.P., calculate the resistance per ton due to friction. *(P.U.)*

6. An engine of 200 H.P. is taking a train of mass 150 tons up an incline of 1 in 224 and the resistance is 5 lb. wt. per ton mass. What is the maximum uniform speed of the train in m.p.h.?

7. A train of mass M lbs. is ascending a smooth incline of 1 in n and when the velocity of the train is v ft. per second, its acceleration is f ft./sec.^{2}, find the effective horse-power of the engine. *(J. & K.)*

8. A locomotive engine developing its maximum horse power of 500 and weighing with the train 200 ton wt. is travelling horizontally against a constant resistance of 3000 lb. wt. Find the acceleration of the train when its speed is 25 miles per hour. Find also its greatest possible speed up a slope of 1 in 30. *(P.U.)*

9. A locomotive engine draws a load of m lbs. up an incline of inclination α to the horizon, the coefficient of friction being μ. If, starting from rest and moving with a uniform acceleration, it acquires a velocity v in t sec., show that the average horse-power at which the engine has worked is

$$\frac{mv}{1100}\left(\frac{v}{gt}+\mu\cos\alpha+\sin\alpha\right).$$

10. Find the H.P. of an engine which can travel at the rate of 25 miles per hour up an incline of 1 in 100, the mass of the engine and load being 10 tons and the resistance due to friction etc., being 10 lbs. weight per ton. *(P.U. 1968)*

11. An engine of mass 30 tons pulls after it a train of mass 130 tons. Calculate the force exerted by the engine if at the end of first mile from the start the speed be raised to 45 miles per hour. *(P.U. 1965)*

(B) Work and Energy

8·4 Energy. *The capacity of a body to do work is known as* **Energy.**

A body may possess energy owing to a variety of causes. Thus the energy present in a railway engine is due to heat or electricity. This energy is further transformed into **mechanical energy** which is the object of our study in Dynamics.

Mechanical Energy is of two types :

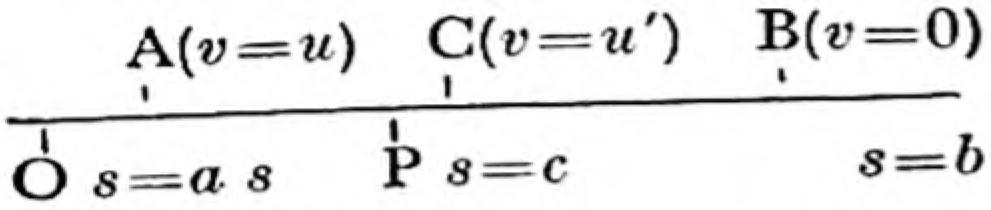
(*i*) *Kinetic Energy* (*K.E.*) *is defined as the capacity of a body to do work in virtue of its motion and is measured by the amount of work which the impressed force will have to do in bringing it to rest.*

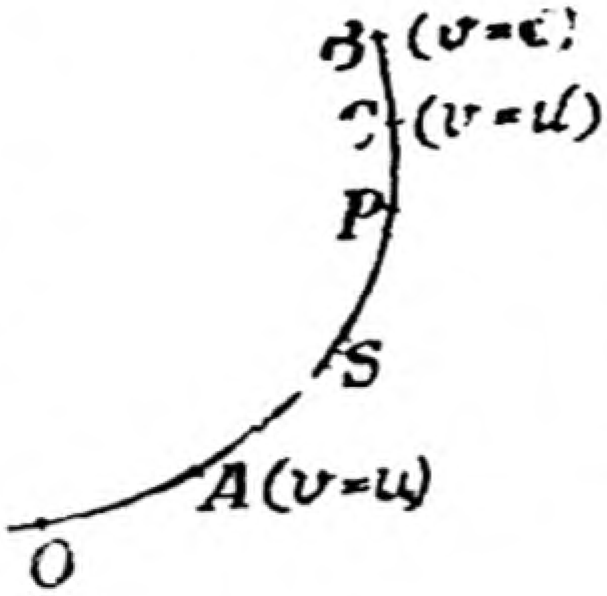
(*ii*) *Potential Energy* (*P.E.*) *is defined as the capacity of a body to do work in virtue of its position and is measured by the amount of work that the impressed force will have to do in bringing it back to its initial position* (known as **Standard Position**). For Examples see Art. 8·43.

Note. Obviously the units for Energy are same as those for work.

8·41 Measure of K.E. of a mass m, moving with velocity u.

Let a mass m be moving along a st. line OX (or a curve) with velocity v at any instant t. Then if its distance measured from a fixed origin on the line (or the curve) be s; *its* velocity is $\dfrac{ds}{dt}$ and acceleration is $\dfrac{d^2s}{dt^2}$ or $v\dfrac{dv}{ds}$.





Consequently the force F, which acts opposite to the direction of motion, in order to bring it to rest is given by

$$F = -mv\frac{dv}{ds}. \qquad \ldots(\text{I})$$

Let velocity at A be u and that at B be zero.

Let $\qquad OA = a$ and $OB = b$.

Now, by definition, K.E. at A is equal to the work done by the impressed force F from A to B, that is till the mass is reduced to rest.

$$\therefore \qquad \text{K.E.} = \int_a^b F.ds. = \int_a^b -mv\frac{dv}{ds}\,.\,ds$$

$$= -m\int_a^0 v\,dv = \tfrac{1}{2}mu^2.$$

Thus K.E. of a mass m moving with a velocity v, in a st. line or in a curved path, is given by ½ mu² and is measured in **Absolute Units** (*because the force F is measured in* **Absolute Units**).

Note. 1. K.E. depends upon mass and the velocity of the mass and is independent of the path or the direction and the nature of forces which bring it to rest.

2. Impressed force is measured by the rate of change of K.E. Rate of change of K.E. relative to distance

$$= \frac{d}{ds}\ (\text{K.E.})$$

$$= \frac{d}{ds}\left(\frac{1}{2}mv^2\right)$$

$$= mv\frac{dv}{ds}$$

$$= mf$$

$= \text{Impressed Force.}$ *(P.U. 1953)*

8·42 Principle of Conservation of Work and Energy.

Statement. *The change in K.E. in any displacement of a particle is equal to the work done by the forces which produce that change.*

Referring to the Figures of Art. 8·41, if the velocity at a point C($s=c$) in the path during motion is u' then the work done from A to C as the particle moves in **the direction of the force F**

$$= \int_a^c \mathrm{F}ds = \int_a^c mv\frac{dv}{ds}ds = m\int_u^{u'} vdv = \frac{1}{2}mu'^2 - \frac{1}{2}mu^2$$

= **Change in K.E.**

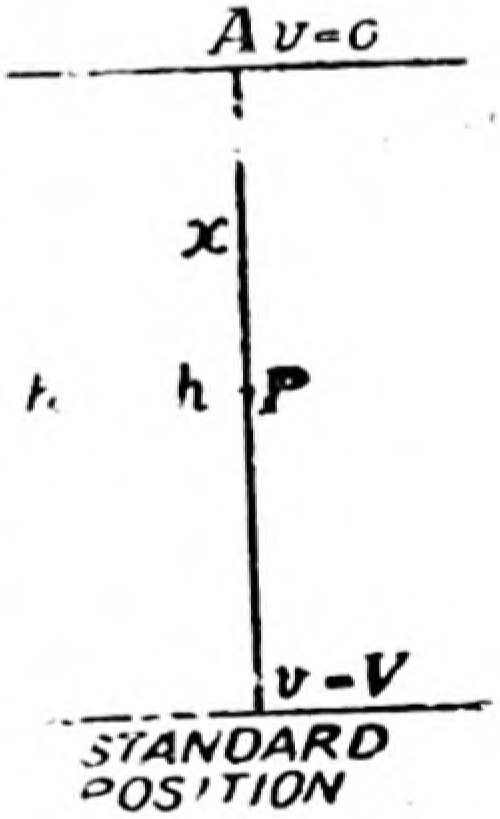
8·43 The **Potential Energy**, *as defined above, of a body is the amount of work it can do in moving from its actual position to some standard position.*

Examples of potential energy are :—A body raised to a height above the ground, a clock when it is wound up, water stored up in a reservoir. Compressed air has potential energy, *viz.*, the work it can do in expanding to the volume it would occupy in the atmosphere.

8·44 Principle of Conservation of Energy

A particle of mass m falls from rest at a height h above the ground. Show that the sum of its potential and kinetic energies is constant throughout the motion. (*P.U. 1958 S*)

The Potential energy at a height h is the work the particle can do in falling to the ground and this is equal to the work done in raising it to a height h, *viz.*, mgh.



$$\text{K.E. at A} = 0 \qquad \ldots\text{(I)}$$

$$\therefore \text{ Sum of K.E. and P.E. at A} = mgh. \qquad \ldots\text{(A)}$$

Let v be the velocity of the particle when it has fallen through a distance x to a point P then

$$v^2 = 2gx. \qquad \ldots\text{(II)}$$

$$\text{Its kinetic energy at P} = \tfrac{1}{2}mv^2 = mgx. \quad \ldots\text{(III)}$$

$$\text{Its potential energy at P} = mg(h-x) \qquad \ldots\text{(IV)}$$

$$\therefore \text{ Sum of K.E. and P.E. at P}$$

$$= mgx + mg(h-x) = mgh. \qquad \ldots\text{(B)}$$

On reaching the ground the velocity V is given by

$$\mathrm{V}^2 = 2gh\ ; \qquad \ldots\text{(V)}$$

$$\text{P.E. at B} = 0. \qquad \ldots\text{(VI)}$$

$$\text{K.E. at B} = \tfrac{1}{2}m\mathrm{V}^2 + mgh.$$

$\therefore$ Sum of K.E. and P.E. at $B = mgh$. ...(C)

From (A), (B) and (C) ; theorem follows.

In fact on reaching the ground, the potential energy has been transformed into kinetic energy.

8·45 The example of the previous article is a simple illustration of *the Principle of the* **Conservation of Energy** which may be stated as follows :

If a body (or system of bodies) be in motion under a conservative system of forces, the sum of its kinetic and potential energies is constant.

Def. *Forces, of the kind which occur in the material universe, are said to be conservative when they depend on the position or configuration only of the system of bodies and not on the velocity or direction of motion of the bodies.*

Thus from a conservative system are excluded forces of the nature of friction or forces such as the resistance of the air which varies as some power of the velocity.

In most cases dealt with in Dynamics the conservative system of forces are those due to gravity. Other examples of conservative forces are (*i*) the attraction between two particles which is a function of their mutual distance (*ii*) any force which acts towards a fixed point and is a definite function of the distance from that point. The principle of energy is most commonly used when considering motion under gravity ; it tells us that in the absence of friction and impacts, for any loss of kinetic energy there must be an equal gain in potential energy and *vice versa*.

Ex. Thus for a body sliding down a smooth inclined plane the kinetic energy acquired is equal to the loss of potential energy.

The kinetic energy, and, therefore, the velocity acquired in sliding down the plane is the same as that acquired by falling vertically through the height of the plane. Similarly for a particle sliding down any smooth curve the velocity at the bottom depends only on the vertical height descended.

Caution. *This principle should never be used in problems where there is any friction or any sudden jerk or impact.*

Ex. 6. *Find the kinetic energy acquired by a wt. of kilogram in falling through a metre.*

$$\text{K.E. acquired} = 981000 \times 100 \text{ ergs} = 98{\cdot}1 \times 10^6 \text{ ergs.}$$

Ex. 7. *A stone weighing 4 lbs. falls through 8 ft. What is its kinetic energy and what force will stop it in 2 ft ?*

$$\begin{aligned}\text{Kinetic energy} &= \text{work done by gravity}\\ &= \text{weight} \times \text{distance fallen}\\ &= 4 \times 8 \text{ ft. lbs.} = 32 \text{ ft. lbs.}\end{aligned}$$

Let P lbs. be the force required to stop it in 2 ft. Then a force of P lbs. acts upwards and the weight 4 lbs. acts downwards. Hence the upward force retarding the motion of the stone is

$$(P-4) \text{ lb. wt.}$$

When the stone is brought to rest, work done against retarding force = Loss in K.E.

$$\therefore \quad (P-4)\times 2=32.$$

$$\therefore \quad P=20 \text{ lb. wt.}$$

Ex. 8. *A mass of 12 lbs. is attached to one end of an elastic string of natural length 4 feet whose other end is fixed at a point A. The modulus of the string is such that 12 lb. mass hanging vertically would stretch the string 6 inches. The mass is held at A and allowed to fall vertically. How far below A will it come to rest ?*

Since 12 lb. wt. stretches the string $\frac{1}{2}$ foot

$$12g=\frac{\lambda}{4}\cdot\frac{1}{2} \qquad \therefore \quad \lambda=96g \text{ poundals.}$$

Initially the mass is at rest and finally it comes to rest. $\therefore$ the kinetic energy is zero initially and in the end. Hence the change in the kinetic energy is zero.

By the principle of work and energy the work done in falling a distance x is 0.

The work done by the wt. $=-12gx$ ft. poundals.

The work done by the string in stretching to a length x

$$=-\frac{1}{2}\,\frac{96g}{4}\,(x-4)^2=-12g(x-4)^2.$$

As the total work done = 0

$$12gx-12g(x-4)^2=0$$

or

$$x-(x^2-8x+16)=0$$

$$\therefore \quad x^2-9x+16=0$$

$$\therefore \quad x=\frac{9+\sqrt{81-64}}{2}$$

$$=\frac{9+\sqrt{17}}{2}=6{\cdot}56 \text{ ft.}$$

The root with the negative sign before the radical can be ignored as in that case x is less than the natural length of the string.

Ex. 9. *Find the velocity of a 10 lb. shot that will just pass through a plate 9 inches thick the resistance being 84 tons.*

Let u be the velocity of the shot in feet per sec.

Resistance $=84\times 2240\times 32$ poundals.

The equation of work is

change in Kinetic Energy = the work done.

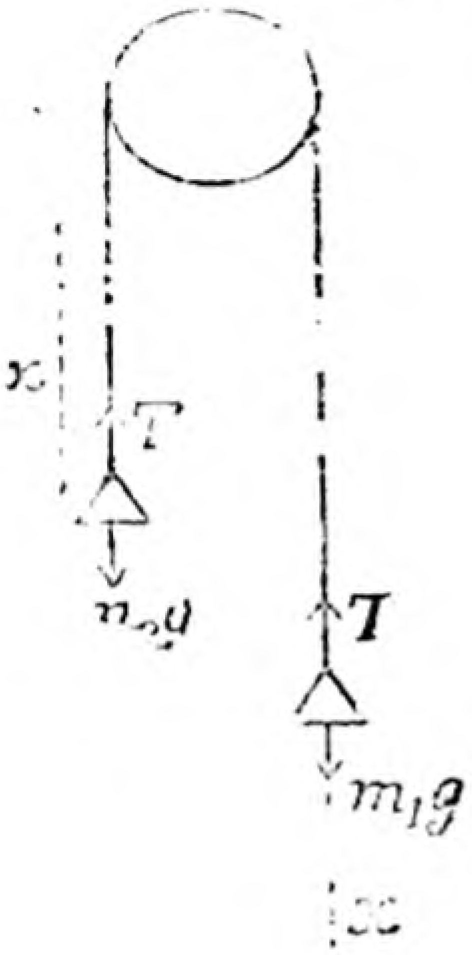
$$\therefore \quad \tfrac{1}{2}\times 10\times 0-\tfrac{1}{2}\times 10u^2=-84\times 2240\times 32\times \tfrac{9}{12}.$$

$$\therefore \quad u^2=63\times 32\times 448$$

$$u=672\sqrt{2} \text{ ft. per sec.}$$

or

Ex. 10. *Apply the Principle of Energy in finding the acceleration of two particles connected by a string placed over a smooth pulley.* *(P.U.)*



Sol. Let m_1, m_2 be the masses hanging by a string and let x be the distance moved by each so that the velocity

$$=\frac{dx}{dt}.$$

Whence K.E. of the system

$$=\tfrac{1}{2}m_1\left(\frac{dx}{dt}\right)^2+\tfrac{1}{2}m_2\left(\frac{dx}{dt}\right)^2 \qquad \ldots(\text{I})$$

Also work done by gravity

$$=m_1gx-m_2gx. \qquad \ldots(\text{II})$$

Now m_1 moves ↓, the work done by T on m_1 is equal and opposite to the work done by T on m_2 which is moving ↑, therefore that need not be considered.

By Principle of Work and Energy, from I and II

$$\tfrac{1}{2}(m_1+m_2)\left(\frac{dx}{dt}\right)^2=(m_1-m_2)gx. \qquad \ldots(\text{II})$$

Whence on differentiation w.r.t. t

$$\tfrac{1}{2}(m_1+m_2)\,.\,2\,.\,\frac{dx}{dt}\,.\,\frac{d^2x}{dt^2}=(m_1-m_2)g\,.\,\frac{dx}{dt}$$

or that
$$f=\frac{m_1-m_2}{m_1+m_2}\,.\,g.$$

8·36. In problems where a mass loses velocity owing to the action of a retarding force, *e.g.*, a bullet passing through a plank or a train being pulled up by its brakes we can obtain a measure of the retarding force in two ways :

(1) If we know the time during which the force acts and the initial and final velocities u and v of the mass m then if F is the average force

Impulse = Change in Momentum

(Impulse = force × time. See Art. 8·5)

or
$$F\times t=m(u-v).$$

(2) If we know the distance s travelled during the retardation, we can obtain the average force by equating the work done to the loss of kinetic energy, *i.e.*,

$$Fs=\tfrac{1}{2}m(u^2-v^2).$$

It must be clearly understood that the force obtained is an average force, the **time average** or **the space average** respectively.

Ex. 11. *An avalanche weighing 3000 lbs. has fallen through 700 ft. What is its K.E. now and what force will stop it in 200 ft. ?* (*P.U.*)

(*ii*) *What force will stop it in 200 seconds ?*

The velocity V acquired in falling under gravity through 700 ft. is $V=\sqrt{2gs}=\sqrt{2\times32\times700}=80\sqrt{7}$ f.p.s.

$\text{K.E.}=\frac{1}{2}mV^2=\frac{1}{2}\times3000\times6400\times7$ ft. lbs. $=187\cdot5$ ft. tons.

By Principle of Work and Energy, if F is the required upward force

$$(F-mg)s=\tfrac{1}{2}mu^2-\tfrac{1}{2}m\,a^2. \qquad \ldots(I)$$

or

$$(F-3000\,g)200=\tfrac{1}{2}\times3000\times6400\times7$$

$$\therefore \qquad F=13500 \text{ lbs. wt.}$$

(*ii*) If P is the required force, its impulse $(P-mg)t$ is equal to change of momentum $=m(0-80\sqrt{7})$

$$(P-3000\,g)\;200=3000\times80\sqrt{7}$$

or

$$P=\left(3000+\frac{75\sqrt{7}}{2}\right)\text{ lbs. wt.}$$

The reader should note the difference in the values of F and P.

Exercise VIII C

1. A body of mass 5 lbs. is projected vertically upwards with a velocity of 40 ft. per second. What is its kinetic energy (*i*) at the moment of projection, (*ii*) after 1 second ? (*iii*) Deduce the distance travelled in 1 sec by Principle of Work and Energy.

2. What is the potential energy of 6000 gallons of water which has been raised 50 ft. above the ground ? (1 gallon of water weighs 10 lbs.)

3. Determine the horse power which could be obtained from the kinetic energy of a stream at a place where the width is 100 ft., the mean depth 20 ft. and the mean velocity $4\frac{1}{2}$ miles per hour.

[1 cubic ft. of water weighs 62·5 lbs.]

4. A steamer of 22000 H.P. makes a run of 3300 miles in 10 days. Find the average resistance to the motion.

5. A labourer has to supply bricks to a bricklayer vertically above him, at a height of 12 feet. He throws them up so that they reach the bricklayer with a velocity of 12 f.p.s. What proportion of work could he save if he threw them so that they might just reach the brick-layer ? *(P.U.)*

6. What is the kinetic energy of a bullet weighing 3 lbs. when moving with a velocity of 30 ft. per second ? The bullet loses a velocity of 5 ft. per second in passing through a plank. How many such uniform planks will it pass through completely before coming to rest, resistance of the planks is supposed to be uniform ?

7. A bullet passes through two planks in succession. Its original velocity is 1200 ft. per sec. and it loses a velocity of 200 ft. per sec. in penetrating each plank. Find the ratio of the thickness of two planks, assuming that they offer the same average resistance. *(P.U.)*

8. A mass of 4 lbs. falls vertically through 100 feet from rest and is then brought to rest by penetrating into sand. Find the mean force exerted by the sand if it stops the mass (*i*) in 2 seconds (*ii*) in 2 ft. *(P.U.)*

9. What is the kinetic energy of a bullet weighing $\frac{1}{32}$ lb. and travelling with a speed of 1200 f.p.s. ? What fraction of its kinetic energy has it lost when passing through a fixed block 15 inches thick which offers an average resistance of 400 lbs. weight ? What thickness of the next similar block will it pierce before coming to rest ? *(P.U.)*

10. A truck weighing 15 tons and travelling at 6 feet per second runs into a spring buffer which compresses half an inch per ton. Prove that the maximum compression of the spring buffer will be nearly 10 inches. *(P.U.)*

11. A steam hammer, of mass 20 tons, falls vertically through 5 ft., being pressed downwards by steam pressure equal to the weight of 30 tons, what velocity will it acquire, and how many ft. lbs. of work will it do before coming to rest ? *(P.U.)*

Further Examples on Principle of Energy

8·46 Ex. 12. *Two particles moving in straight lines, the first acted on by a constant force, the second acted on by a force doing work at a constant rate, each have their velocities increased from V to 2V after travelling a distance a. Show that the time taken by the second is $\frac{27}{28}$ of that taken by the first.*

Let f be the constant acceleration of the first particle then

$$4V^2 = V^2 + 2fa$$

$$\therefore \qquad f = \frac{3V^2}{2a}.$$

If t_1 be the time taken

$$2V = V + ft_1$$

$$\therefore \qquad t = \frac{V}{f} = \frac{2a}{3V}.$$

In the case of the second if P be the force, v the speed and H the rate of working,

$$pv = H.$$

or $$p=\frac{H}{v}\,.$$

Now $$mv\frac{dv}{ds}=p=\frac{H}{v}\,.$$

$\therefore$ $$v^2dv=\frac{H}{m}ds.$$

Integrating $$\frac{v^3}{3}=\frac{H}{m}s+C$$

when $$s=0,\ v=V.$$

$\therefore$ $$C=\frac{V^3}{3}$$

when $$s=a,\quad v=2V$$

$$\frac{8V^3}{3}=\frac{Ha}{m}+\frac{V^3}{3}\,.$$

$\therefore$ $$\frac{H}{m}=\frac{7V^3}{3a}\,.$$

Also $$m\frac{dv}{dt}=\frac{H}{v}$$

$$vdv=\frac{H}{m}dt.$$

$\therefore$ $$\frac{v^2}{2}=\frac{H}{m}t+C$$

when $$t=0,\quad v=V,\quad C=\frac{V^2}{2}$$

when $$t=t_2,\quad v=2V$$

$$2V^2=\frac{H}{m}\,t_2+\frac{V^2}{2}\,.$$

$\therefore$ $$\frac{H}{m}\ t_2=\frac{3V^2}{2}$$

or $$t_2=\frac{2V^2}{2}\times\frac{3a}{7V^3}\,.$$

$\therefore$ $$t_2=\frac{9a}{14V}\,.$$

$\therefore$ $$\frac{t_2}{t_1}=\frac{9a}{14V}\cdot\frac{3V}{2a}=\frac{27}{28}\,.$$

Exercise VIII (D)

1. In starting a train the pull of the engine on the rails is at first constant and equals P ; after the speed attains a certain value u, the engine works at a constant rate $R=Pu$. Prove that when the engine has attained a speed $v > u$, the time t and distance x from the start are given by

$$t=\frac{1}{2}\ \frac{\text{M}}{\text{R}g}\ (u^2+v^2),\ \ x=\frac{1}{3}\ \frac{\text{M}}{\text{R}g}\left(\text{V}^3+\frac{u^3}{3}\right).$$

(P.U., 1961)

2. A horse pulls a wagon of 10 tons from rest against a constant resistance of 50 lb. The pull exerted is at first 200 lb. and decreases uniformly with the distance until it falls to 60 lb. after a distance of 167 feet has been covered. Show that the resultant velocity of the wagon is very nearly 6 ft./sec.

3. A mass m is attached to one end of an elastic string of natural length a, the other end being fixed to a peg. Initially m is held near the peg and projected with the velocity v vertically downwards. If it moves all along in a st. line, the depth h below the peg, of mass m when it is first at rest is given by the equation

$$v^2+2gh=\frac{\lambda}{ma}\ (h-a)^2,$$

where λ is the modulus of elasticity. ***(Bombay)***

4. The resistance to the motion of train is 160 lb. wt. per ton mass at all speeds. It is moving on the level with uniform speed V and comes to an incline of 1 into 70. The engine continues to work at the same rate as before. Prove that if v is the velocity and x the distance described up the incline in time t then

$$v^2=\text{V}^2-\frac{g}{35}\ (6x-5\text{V}t).$$

5. A train of weight M lb. moving at v ft./sec. on the level is pulled with a force of P lb. against a resistance of R lb. Show that in accelerating from v_0 to v_1 ft./sec. the distance in feet described by the train is

$$\frac{\text{M}}{g}\int_{v_0}^{v_1}\frac{v\,dv}{\text{P}-\text{R}}.$$

6. A ring of mass m slides on a smooth vertical rod, attached to the ring is a slight string passing over a smooth peg distant a from the rod and the other end of this string is a mass $\text{M}(>m)$. The ring is held on a level with the peg and released. Show that it first comes to rest after falling a distance $\dfrac{2m\text{M}a}{\text{M}^2-m^2}$.

7. A light string ABCDE whose middle point is C passes through smooth rings B, D which are fixed in a horizontal plane at a distance $2a$ apart. To each of the points A, C, E is attached a mass m. Initially C is held at rest at O, the middle point of BD and is then set free. Show that C will come to instantaneous rest when

$$\text{OC}=\frac{4a}{3}.$$

Show also that when C has fallen through $\frac{3a}{4}$ from O its velocity is

$$\sqrt{\frac{25ag}{86}}.$$

(C) Impulse and Energy

8·5 Impulse of a constant force in a given time *is equal to the product of the force and the time t during which it acts,*

i.e., $$I = Pt \qquad \ldots(I)$$

Thus when **P is constant**

$$Pt = mft = m(v-u)$$

(ii) when P is Variable, Impulse is defined as the integral of the force with respect to time

i.e., $$I = \int_0^t P dt = \int_0^t m \frac{dv}{dt} dt$$
$$= \int_0^t m dv = m(v-u).$$

Hence in both cases :

Impulsive of force = Change of momentum produced

8·51 Impulse Forces or Blows.

Suppose the force P is very large but acts only for a very short time. The body will move a very short distance whilst the force is acting, so that the change of position of the body may be neglected. The total effect of the force is measured by the Impulse or the Change of momentum it produces. Such a force is called an impulsive force. Hence

An impulsive force is a very great force acting for a very short time so that the change in the position of the particle during the time the force acts on it may be neglected.

Caution. "Impulse" is different from "Impulsive Force".

These two terms may not be confused.

An impulsive force or a blow brings in a sudden change of motion. Instantaneous changes of velocities take place. The effect of an *Impulsive Force* is measured by *change of momentum* (*or impulse*) produced by it.

Examples of Impulse Forces are : the blow of a hammer, the impact of a bullet on a target and the collision of two billiard balls.

8·6 Principle of Conservation of Linear Momentum.

When two bodies A and B impinge then by the third law of motion the action of A on B is at any instant during their contact equal and opposite to that of B on A.

Hence the impulse of the action of A on B is equal and opposite the impulse of the action of B on A.

It follows that the change in the momentum of B is equal and opposite to the change in the momentum of A and therefore the sum of these changes measured in the same direction is zero.

Hence the sum of the momenta of the two masses measured in the same direction is unaltered by their impact. ***(Panjab, 1958 S)***

8·61 Applications

If a mass m_1 moving with velocity v strike a mass m_2 which is free to move in the direction of the motion of m_1 and the two move on as a single inseparable body, there is no loss of momentum. If V be the common velocity of the two together after the impact

$$(m_1+m_2)V = m_1v + m_2 \times 0 = m_1v$$

$$\therefore \qquad V = \frac{m_1v}{m_1+m_2}. \qquad \text{...(I)}$$

$$\text{K.E. before impact} = \tfrac{1}{2}\, m_1v^2 + \tfrac{1}{2}m_2.0^2 = \tfrac{1}{2}mv^2 \qquad \text{...(II)}$$

$$\text{K.E. after impact} = \frac{1}{2}(m_1+m_2)\,V^2 = \frac{1}{2}\left(\frac{m_1v}{m_1+m_2}\right)^2 \qquad \text{...(III)}$$

and this is obviously less than $\frac{1}{2}mv^2$.

Observation. Although there is no loss of monentum there is loss of kinetic energy.

Caution. *It is important to realise that the principle of momentum can only be applied in a direction in which there is no external impulse force acting.*

8·62 Motion of a Shot and a Gun

When a gun is fired the powder is almost instantaneously converted into a gas at a very high pressure which by its expansion forces the shot out. The force exerted on the shot forwards is, at any instant before the shot leaves the gun, equal and opposite to that exerted by the shot on the gun backwards and, therefore, the impulse of this force on the shot is equal and opposite to the impulse of the force on the gun. Hence the momentum generated in the shot is equal and opposite to that generated in the gun, if the latter be free to move. Surely then the gun will recoil.

If, however, the gun is elevated, the momentum of the gun is not equal and opposite to that of the shot.

The horizontal momentum of the gun is equal to the horizontal momentum of the shot but any vertical momentum imparted to the gun is at once destroyed by the impulsive pressure of the plane on which it stands.

Any apparatus for preventing the horizontal recoil of the gun does not introduce an impulsive force at the instant of firing.

8·63 Impact of water on surface

When a jet of water impinges against a fixed surface as in the case of the hose pipe of a fire engine or a continuous downpour of rain we have a series of successive impacts. The amount of momentum destroyed per second can be calculated and thus gives us the average force on the surface.

Let v be the velocity of water increasing from the nozzle of a pipe held at right angles to a fixed surface. Assuming r to be radius of the nozzle and m the mass of unit volume of water, the mass of water issuing from the nozzle in unit second is $mv\pi r^2$ and its momentum is $mv^2\pi r^2$. If there be no splash of water after striking the surface, this momentum of water per second is destroyed by the impulse of pressure of the wall. Using the result

Impulse = change in momentum

we have

$$Pt = mv^2\pi r^2$$

Since $t = 1$,

$$\therefore \quad P = mv^2\pi r^2$$

which gives pressure due to jet water impinging against the fixed surface.

Example 13. *A body of mass 3 lbs. moving with velocity 13 feet per second overtakes a body of mass 2 lbs. moving with velocity 3 feet per second in the same straight line and they coalesce and form one body, Find the velocity of this single body.*

Let V be the required velocity, then since the sum of the moments of the two bodies is unaltered by impact, we have

$$(3+2)V = 3\times 13 + 2\times 3 = 45$$

$$\therefore \quad V = 9 \text{ ft. per second.}$$

Example 14. *A shot whose mass is 400 lbs. is projected from a gun of 50 tons with a velocity of 900 feet per second. Find the resulting velocity of the gun.*

Since the momentum of the gun is equal and opposite to that of the shot, we have

if v be the velocity communicated to the gun

$$50\times 2240v = 400\times 900$$

$$\therefore \quad v = 3\tfrac{3}{14} \text{ ft. per sec.}$$

Example 15. *Water issues from a circular pipe of 3 inches diameter with a velocity of 150 ft. per sec.; find the weight of water discharged per minute. If the water impinges directly upon a wall and its momentum is thereby wholly destroyed, what is the pressure of the jet upon the plane?*

The area of the cross-section of the pipe is $\frac{\pi}{4}\times\frac{1}{16}$ square feet and a column 15 ft. long is discharged every second.

The volume discharged per minute is

$$\frac{\pi}{64}\times 15\times 60 \text{ cu. feet.}$$

and its $\text{mass}=\dfrac{\pi\times 15\times 60\times 125}{64\times 2}$ lbs. [mass of 1 cu. ft. of water=62·5 lbs]

$$=2762 \text{ lbs. nearly.}$$

The momentum destroyed per second

$$=\frac{\pi\times 15\times 15\times 125}{128} \text{ ft. lb. units.}$$

The $\text{pressure}=\dfrac{\pi\times 225\times 125}{128}$ poundals [$Pt=mv$, $\because\ t=1$ $\therefore\ P=mv$]

$$=21\cdot 58 \text{ lb. wt.}$$

Exercise VIII (E)

1. A bullet of mass 2 oz. is fired into a target with a velocity of 1280 ft./sec. The mass of the target is 10 lbs. and it is free to move; find the loss of kinetic energy by the impact in ft. poundals. (*P.U.*)

2. A shot of mass m is fired from a gun of mass M with velocity u relative to the gun; show that the actual velocities of the shot and gun are $\dfrac{Mu}{M+m}$ and $\dfrac{mu}{M+m}$ respectively and that their kinetic energies are inversely proportional to their masses. Show also that the total kinetic energy generated is

$$\frac{1}{2}\,\frac{Mm}{M+m}\cdot u^2$$

3. Prove that the sum of the momenta of two masses, in any given direction, is unaltered by their mutual impulsive action.

A gun of mass 5 tons discharges a shot of mass 112 lbs. horizontally. The recoil of the gun is opposed by a constant force of 2 tons wt. which brings it to rest in $1\frac{1}{4}$ seconds. Find the velocity of the shot, and also the energy of the explosion in foot tons. (*P.U.*)

4. A ball of 2 oz. in weight falls from a height of 36 feet and rebounds to a height of 25 feet. Find the impulse and the average force between the ball and the floor, if the duration of the impact is $\frac{1}{100}$ seconds. (*P.U.*)

5. A jet of water of circular cross-section of diameter 2 inches and velocity 36 f.p.s. impinges normally upon a plane inelastic wall, so that the velocity of the water is destroyed on reaching the wall. Calculate the thrust on the wall. (*P.U. B.Sc., 1960 S*)

6. A jet of water leaves a nozzle of 1 inch diameter at a speed of 50 ft. per second and impinges on a plate fixed at right angles to its direction. What pressure is exerted on the plate?

If the nozzle is drawn back with a velocity of 20 ft. per second in the line of the jet, what pressure is then exerted on the plate? *(P.U.)*

7. A gun of mass M fires a shell of mass m horizontally and the energy of the explosion is such as would be sufficient to project the shell vertically to a height h. Show that the velocity of recoil of the gun is

$$\left[\frac{2\,m^2gh}{M(M+m)}\right]^{\frac{1}{2}}. \qquad (P.U.)$$

8. A body of mass m_1+m_2 is split into two parts of masses m_1 and m_2 by an internal explosion which generates kinetic energy E. Show that if after explosion the parts move in the same line as before their relative speed is

$$\sqrt{\frac{2E(m_1+m_2)}{m_1m_2}}. \qquad (P.U.)$$

9. A shell of mass M is moving with velocity V in the line AB. An internal explosion which generates an energy E breaks it into two masses whose ratio is $m_1 : m_2$, which continue to move in the line AB. Show that their velocities are

$$V+\sqrt{\frac{2m_2E}{m_1M}},\ V-\sqrt{\frac{2m_1E}{m_2M}}. \qquad (P.U.\ 1960\ S)$$

10. A bullet of mass m moving with velocity v, strikes a block of mass M, which is free to move in the direction of the bullet and is embedded in it. Show that the loss of kinetic energy is

$$\frac{M\,mv^2}{2(M+m)}.$$

If the block is afterwards struck by an equal bullet moving in the same direction with the same velocity, show that there is a further loss of energy equal to

$$\frac{M^2\,m\,v^2}{2(M+2m)(M+m)}.$$

11. A hammer of mass M gms falls from a height h cm upon the top of a pile of mass m gms and drives it into the ground a distance a cm. Find the resistance of the ground it being assumed to be constant and the pile being supposed inelastic.

12. Prove that if a hammer of mass M lb. striking a nail of m lb. with velocity u feet per second drives it a feet into block of wood of mass m_1 lb; the average resistance of the wood in pounds to the penetration is $\frac{M^2}{M+m}\,\frac{u^2}{2ga}$. In case the block is free to recoil, the resistance obtained will be $\frac{m_1\,M^2}{(M+m+m_1)\,(M+m)}\,\frac{u^2}{2ga}$. *(P.U. 1969)*

Miscellaneous Exercise VII

1. An engine gets up a velocity of 30 m.p.h. in one minute and 50 seconds from rest in a train of 200 tons wt. on a level line. If the resistances are 14 lb. wt. per ton and the velocity is uniformly accelerated, find the greatest horse power the engine is developing and also the greatest velocity it can get up if this horse power is maintained. *(P.U., 1955)*

2. A car is travelling at its maximum speed of 45 m.p.h. on the level, the resistance being 100 lb. wt. per ton assumed to be independent of the speed. It then climbs a hill of gradient 1 in 20 and the speed falls until it is steady, the engine then working at the same effective horse power as before. Find its steady speed up hill. *(P.U., 1954)*

3. An electric crane raises a load of 5 tons through a height of 40 feet in 14 seconds at a uniform speed. Assuming that 25 per cent of the power supplied is wasted in friction, what is the H.P. that has to be supplied to the motor.

4. A motor car weighing 1 ton attains a maximum speed of 32 miles an hour when running freely down a slope of 1 in 10. What H.P. must the engine develop to take it up an incline of 1 in 20 at this speed ? Why is it impossible to calculate the H.P. at any other speed ?

5. A ton of coal is raised to the surface in a cage whose weight is 6 cwt. 88 lbs. from the bottom of a mine 330 feet deep by a rope of wt. 2 lb. per foot. Find the work done and the H.P. of an engine which can do it in a minute.

6. A cyclist whose weight added to that of the machine is 200 lbs., is riding on a level road at the rate of 10 miles an hour, his bicycle is geared upto 70 inches and the length of the cranks is 7 inches ; if the resistance to the motion be 5 lbs. wt., find the downward thrust he must exert on his pedals and the rate at which he works compared with a horse power.

(**Note**. "Geared up to 70 inches" means that for every revolution of the rider's feet the bicycle advances $\pi.70$ inches).

7. A constant force of P lb. wt. is applied vertically upwards to a body of mass M lb. for a certain time. It then ceases to act and the body continues to move upwards to reach its destination in a total time t seconds. Prove that the greatest horse power developed by the force of P lb. wt. is $(P-M)gt/500$.

8. A shell of mass m is fired from a gun of mass M which can recoil freely on a horizontal base and the elevation of the gun is α. Prove that the initial inclination of the path of shell to the horizon is

$$\tan^{-1}\left[\left(1+\frac{m}{M}\right)\tan\alpha\right].$$

9. A shell of mass m is ejected from a gun of mass M by an explosion which generates kinetic energy E. Prove that the initial velocity of the shell is

$$\sqrt{\frac{2ME}{(M+m)m}},$$

assuming that the gun is free to recoil at the instant of explosion.

10. A shell of mass M is moving. An internal explosion generates an amount E of energy and thereby breaks the shell into masses whose ratio is $m_1 : m_2$. The fragments continue to move in the original direction of the shell. Show that the velocity of one fragment relative to the other is

$$\sqrt{\frac{2E}{M}}\,\frac{m_1+m_2}{\sqrt{m_1m_2}}$$ (*P.U., 1956 S*)

11. A mass m after falling freely through a feet begins to raise a mass M greater than itself and connected with it by means of an inextensible string passing over smooth fixed pulley. Show that M will have returned to its original position at the end of time

$$\frac{2m}{M-m}\sqrt{\frac{2a}{g}}.$$ (*P.U., 1958 S*)

12. A gun is mounted on a gun carriage movable on a smooth horizontal plane, and the gun is elevated at an angle α to the horizon ; a shot is fired and leaves the gun in a direction inclined at an angle θ to the horizon. If the mass of the gun and its carriage be n times that of the shot, show that

$$\tan\theta=\left(1+\frac{1}{n}\right)\tan\alpha.$$

13. A uniform string of mass M and length $2a$ is placed symmetrically over a smooth peg and has particles of m, m' attached to its ends. Show by the Principle of Energy that when the string runs off the peg, its velocity is

$$\sqrt{\frac{M+2(m-m')}{M+m+m'}}.$$ (*P.U.*)

14. If the resistance $R=a+bv^2$, find an expression for the distance described when the power P is shut off and the velocity decreases from v_1 to v_0.

15. The resistance to an aeroplane when landing is $a+bv^2$ per unit mass, v being the speed, a, b constants. For a particular machine $b=10^{-3}$ ft. lb. per sec. units and it is found if the landing speed is 50 m p h., the length of the run before coming to rest is 150 yards. Calculate the value of the constant a.

ANSWERS

Exercise VIII (A), Page 178

2. 6 ft. lbs. **5.** 45144000 ft. lbs. ; $20520\sqrt{10}$ ft. lbs. per sec.

6. 6 ft. lbs. **8.** 1614000 ft. lbs.

Exercise VIII (B), Page 180

1. H.P. 320 **2.** $24\frac{47}{112}$ tons.

3. 2 poundals ; 450 ft. poundals ; Max. H.P.$=\frac{3}{880}$.

4. $\frac{61}{280}$ f.p. sec^2. **5.** 24·06 lbs. per ton.

6. $33\frac{1}{3}$ m.p.h. **7.** $\frac{Mv\,(mf+g)}{550\,ng}$.

8. $\frac{9}{28}$; V$=10{\cdot}4$ m.p.h. approx.

9. 12450 lbs. wt. ; H.P.$=905\frac{5}{16}$.

10. H.P.$=21\frac{3}{5}$. **11.** $2\frac{1}{16}$ tons wt.

Exercise VIII (C), Page 187

1. (*i*) 125 ft. lbs. (*ii*) 5 ft. lbs. (*iii*) 124 ft.

2. 3000000 ft. lbs. **3.** 1020·94 H.P.

4. $267\frac{6}{7}$ tons wt. **5.** 3 : 19

6. 1350 ft. lbs , 3 completely.

7. 11 : 9 **8.** 9 lbs. ; 204 lbs.

9. $703\frac{1}{8}$ ft. lbs. ; $\frac{32}{45}$; $6\frac{2}{32}$ in.

10. $20\sqrt{2}$; 250 ft. tons.

Exercise VIII (E), Page 189

1. $3160\frac{39}{81}$ ft. lbs. **3.** 1600 f.p.s. ; 2020 ft. tons.

4. 11.F.P.S. units ; 1104 poundals.

5. 55·2 lbs. wt. **6.** 26·6 lbs. wt. ; 9·59 lbs.

Miscellaneous Exercise VII, Page 196

1. 672 H.P. ; 90 m.p.h. **2.** $21\frac{12}{53}$ m.p.h.

3. 77·57 H.P. **4.** 286·72 H.P.

5. 333 H.P.

6. P$=39\frac{1}{4}$ lbs. wt. nearly ; Rate $\frac{2}{15}$ H.P.

14. $\frac{1}{gb}\ \log\left[\frac{a+bv_2^2}{a+bv_1^2}\right]$.

CHAPTER IX

IMPACT OR COLLISION OF ELASTIC BODIES

9·1 Elasticity. Elasticity is tendency of a material body to resume its original shape or position after compression or collision.

This property comes to our observation, when we compress a rubber ball and find that as soon as we release it, it resumes its original shape. Again when we drop a glass or ivory or wooden ball from a certain height h on to a hard floor, it rebounds through a certain height h'. It can be easily observed that

(*i*) h' is always less than h.

(*ii*) h' is different for balls of different materials.

Now the velocity of each ball on reaching the floor was same, *viz.*, $v=\sqrt{2gh}$; but the velocity after leaving the floor is naturally different as h' is different for each ball. This shows that there is a variation in the degree of elasticity of each ball depending upon its material. It is denoted by e and is known as the *Coefficient of Elasticity or Restitution.*

The value of e varies from 0 to 1.

For the mutual impact of two glass balls, $e=0·9$ nearly.

For two ivory balls, $e=0·8$ approximately, while for two lead balls, e is about 0·2 only.

There are two hypothetical stages also, which do not exist in reality. They are

$e=0$ for perfectly inelastic bodies, whilst

$e=1$ for perfectly elastic ones.

The name "Inelastic" is given to those bodies which after collision do not separate but move as one single body, *e.g.*, if a railway truck moving with a velocity u strikes against another at rest (or moving), then they move together. But on a close observation we find that at the instant of collision there is a slight restitution just after the compression which shows that the trucks were not perfectly inelastic.

Similarly there are no two bodies perfectly elastic though we deal with them in theory, where we shall take $e=1$ for such bodies.

9·2 Impact or collision of smooth elastic bodies.

Generally we shall consider the bodies as **smooth** spheres so that the mutual action and reaction at the instant of collision is along the normal.

Since the normal passes through the centre in case of a sphere, by considering the spheres to be of same size (though of different masses), the line joining their centres, along which the mutual action and reaction take place, is parallel to the floor. This shall afford us much convenience in dealing with the problems. Impact can be of two types.

Direct. When the direction of motion of each sphere at the instant of collision is along the common normal, the impact is said to be *direct*.

Oblique. When the direction of motion of either or both the spheres is not along the common normal, the impact is said to be *oblique*.

9·21 The whole of the structure of Theory of Impact is based on two principles :

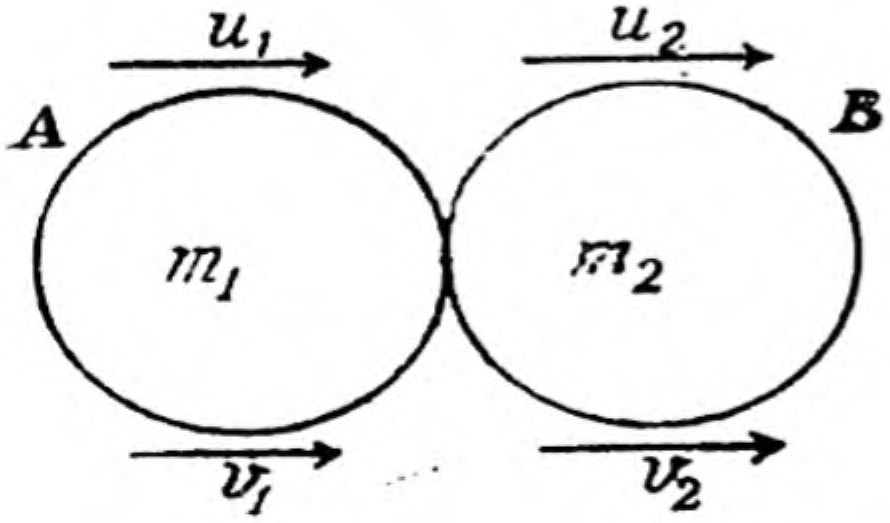
(*i*) *Principle of Conservation of Linear Momentum* (*P.C.M.*) *and* (*ii*) *Newton's Experimental Law* (*N.E.L.*).

(*i*) Principle of Conservation of Linear Momentum has already been established in a previous chapter and states that if in a certain direction there are no external forces then the total linear momentum of the system in that direction remains constant.

Now, in case of the collision of two smooth spheres, the only forces which come into play at the instant of impact are mutual action and reaction which by Newton's III law being equal and opposite cancel each other, hence make it possible for us to apply the Principle of Momentum. Thus if u_1, u_2 are the velocities of two spheres of masses m_1, m_2 before impact and v_1, v_2 their velocities after impact ; then by P.C.M.

Initial momentum, *i.e.*, $m_1u_1+m_2u_2$=Final momentum, *i.e.*,

$$m_1v_1+m_2v_2 \qquad \text{...(I)}$$



(*ii*) *Newton's Experimental Law* (*N.E.L.*) *states that*

When two bodies impinge directly, their relative velocity after impact, is in a constant ratio e to their relative velocity before impact, and is in the opposite direction. The constant ratio e is called the **coefficient of Elasticity on Restitution** (explained in Art. 9·1).

So that
$$\frac{v_1-v_2}{u_1-u_2}=-e \qquad \text{...(II)}$$

In case of oblique impact, the fact contained in N.E.L. holds for the component velocities along the common normal at the instant of impact.

9·3 Direct Impact of two smooth spheres.

(*Delhi, 1956 ; Panjab, 1961 S ; B.Sc , 1960 S*)

Let m_1, m_2 be the masses, u_1, u_2 be the velocities before the impact and v_1, v_2 those after the impact of two smooth spheres moving along the line of centres.

Let e be the coefficient of restitution, then by Newton's Experimental law, their relative velocity after the impact is in the opposite direction of that before impact and bears the ratio e to it ; that is,

$$\frac{v_1-v_2}{u_1-u_2}=-e$$

or

$$v_1-v_2=-e(u_1-u_2) \qquad \text{...(I)}$$

Also by the Principle of Conservation of Momentum

$$m_1v_1+m_2v_2=m_1u_1+m_2u_2 \qquad \text{...(II)}$$

Multiplying (I) by m_2 and adding to (II), we get

$$(m_1+m_2)v_1=(m_1-em_2)u_1+m_2(1+e)u_2 \qquad \text{...(III)}$$

Multiplying (I) by m_1 and subtracting from (II), we get

$$(m_1+m)_2v_2=m_1(1+e)u_1+(m_2-em_1)u_2 \qquad \text{...(IV)}$$

These equations give v_1 and v_2.

Note. It is important to note that u_1, u_2, v_1 and v_2 are assumed *in the same direction* which we may call as positive. If any of them is in the opposite direction, it will have a negative sign affixed to it.

Cor. 1. The impulse of the blow on sphere A = change in the momentum of A

$$=m_1(u_1-v_1)$$

$$=\frac{m_1m_2(1+e)(u_1-u_2)}{m_1+m_2} \qquad \text{...(V)}$$

Cor. 2. If two spheres are *perfectly elastic* and if their *masses are equal, i.e. if* $e=1$ and $m_1=m_2$ then $v_1=u_2$ and $v_2=u_1$.

Hence, *if two equal perfectly elastic balls impinge directly they interchange their velocities.*

Cor. 3. If the two spheres be inelastic

$$e=0,\ v_1=v_2=\frac{m_1u_1+m_2u_2}{m_1+m_2}$$

In fact they coalesce into one body in this case after impact and move together.

9·31 Loss of Kinetic Energy due to direct impact.

(*Panjab, 1962 ; Delhi, 1959*)

We have by Algebra

$$(m_1+m_2)(m_1u_1^2+m_2u_2^2)=(m_1u_1+m_2u_2)^2+m_1m_2(u_1-u_2)^2 \qquad \text{...(I)}$$

and

$$(m_1+m_2)(m_1v_1^2+m_2v_2^2)=(m_1v_1+m_2v_2)^2+m_1m_2(v_1-v_2)^2 \qquad \text{...(II)}$$

Also from Art. 9·3 above

$$m_1v_1+m_2v_2=m_1u_1+m_2u_2 \qquad \text{...(III)}$$

$$v_1-v_2=-e(u_1-u_2) \qquad \text{...(IV)}$$

Subtracting (II) from (I) and using (III), (IV), we get

$$\text{Loss in K E.}=\tfrac{1}{2}(m_1u_1^2+m_2u_2^2)-\tfrac{1}{2}(m_1v_1^2+m_2v_2^2)$$

$$=\tfrac{1}{2}\,\frac{m_1m_2}{m_1+m_2}\,(u_1-u_2)^2(1-e^2).$$

Since $e<1$, this is positive and there is always a loss of K. Energy. This vanishes only when $e=1$, *i.e., when the spheres are perfectly elastic there is no loss of energy.*

Ex. 1. *A ball impinges directly upon another ball at rest, and is itself reduced to rest by the impact ; if half of the initial K.E. is destroyed in the collision ; find the coefficient of restitution.*

(Panjab, B.Sc., 1962)

Let m_1, m_2 be the masses of the two balls. Let u be the initial velocity of m_1, and v the final velocity of m_2. Then by P.C.M.

$$m_1u+m_2\times 0=m_1\times 0+m_2v$$

$$\therefore \qquad m_2v=m_1u \qquad \text{...(I)}$$

Also by N.E.L.,

$$\frac{v-0}{0-u}=-e$$

or

$$v=eu \qquad \text{...(II)}$$

Again, initial K.E. $=\tfrac{1}{2}m_1u^2+\tfrac{1}{2}m_2(0)=\tfrac{1}{2}m_1u^2$

and final K.E. $=\tfrac{1}{2}m_1(0)+\tfrac{1}{2}m_2v^2=\tfrac{1}{2}m_2v^2.$

By Hypothesis, $\tfrac{1}{2}m_2v^2=\tfrac{1}{2}\times\tfrac{1}{2}m_1u^2$

or

$$m_2v^2=\tfrac{1}{2}m_1u^2 \qquad \text{...(III)}$$

Multiplying (I) and (II) and dividing by (III), we get

$$e=\tfrac{1}{2}.$$

Ex. 2. *Two spheres of masses M, m impinge directly when moving in opposite directions with speeds u, v respectively and the sphere of mass m is brought to rest by the collision ; prove that*

$$v(m-eM)=M(1+e)u.$$

After the collision the sphere of mass M is acted on by a constant retarding force which brings it to rest after moving a distance a. Prove that the magnitude of this force is

$$\frac{Me^2(u+v)^2}{2a}.$$

Let the velocities of the two spheres after impact be u' and v' : then by P.C.M.

$$Mu'+mv'=Mu-mv \qquad \text{...(I)}$$

and by N.E.L.

$$u'-v'=-e(u+v). \qquad \text{...(II)}$$

Now $$v'=0 \qquad \text{(Hypothesis)} \qquad \ldots\text{(III)}$$

$\therefore$ $$Mu'=Mu-mv$$

$$u'=-e(u+v).$$

Substituting the value of u', we get

$$-Me(u+v)=Mu-mv.$$

$\therefore$ $$v(m-eM)=M(1+e)u$$

or $$v=\frac{M(1+e)u}{m-eM} \qquad \ldots\text{(IV)}$$

(*ii*) Kinetic Energy of the sphere M

$$=\tfrac{1}{2}Me^2(u+v)^2.$$

If this sphere is brought to rest by a force P in moving a distance a, then by Principle of Work and Energy,

$$Pa=\tfrac{1}{2}Me^2(u+v)^2$$

$\therefore$ $$P=\frac{Me^2(u+v)^2}{2a}.$$

Exercise IX (A)

1. A ball of mass 2 lbs. impinges directly on a ball of mass 1 lb., which is at rest. Find the coefficient of restitution if the velocity with which the larger ball impinges be equal to the velocity of the smaller ball after impact. (*P.U.*)

2. If two inelastic spheres have a direct impact, show that the K.E. lost due to impact is that of a body whose mass is half the harmonic mean between those of the spheres and whose velocity is equal to their relative velocity before impact. (*P.U. B.Sc., 1959*)

[**Hint**. See Art. 9·31]

3. Two masses m and m' are moving in the same straight line, prove that their Kinetic Energy is

$$\frac{1}{2}(m+m')V^2+\frac{1}{2}\frac{mm'}{m+m'}v^2$$

where V is the velocity of their centre of mass and v is their relative velocity.

If there is a direct impact between the masses, prove that their loss of Kinetic energy is

$$\frac{1}{2}\frac{mm'}{m+m'}(1-e^2)v^2$$

where e is the coefficient of restitution. (*Delhi, 1957*)

4. If a ball overtakes a ball of twice its own mass moving with one-seventh of its velocity, and if the coefficient of restitution between them be $\frac{3}{4}$, show that the first ball will, after striking the second ball remain at rest. (*P.U., 1968 S ; Delhi, 51*)

5. A ball A moving with velocity u impinges directly on an equal ball B moving with velocity v in the opposite direction. If A be brought to rest by the impact, show that $u : v = 1+e : 1-e$, where e is the coefficient of restitution. *(P.U.)*

6. A sphere of mass 1 lb. moving at 10 ft./sec. overtakes another sphere of mass 5 lb. moving in the same line at 3 ft/sec. Find the loss of Kinetic Energy during impact and show that the direction of motion of the first sphere is reversed (coefficient of restitution$=0{\cdot}75$).

7. A sphere of mass m, moving with velocity u, impinges directly on an equal sphere at rest. Find their velocities after impact if

(*i*) the coefficient of restitution be $\cdot 5$.

(*ii*) if the spheres are perfectly elastic.

(*iii*) the spheres are inelastic and after impact they coalesce into one body. *(P.U. B.Sc., 1960 S)*

8. Three equal elastic spheres A, B, C are lying in a straight line in this order on a smooth floor. Initially B, C are at rest and not in contact and A is given a velocity u towards B and C. If their coefficient of restitution is $\frac{1}{3}$, find how many impacts will take place and find the ultimate velocities of the three spheres. *(P.U.)*

9. Two equal spheres A, B lie on a smooth horizontal circular groove at opposite ends of a diameter. If A is projected along the groove and at the end of time t it impinges on B, show that the second impact will occur after a further time $\dfrac{2t}{e}$ where e is the coefficient of elasticity. *(P.U., 1960 S)*

10. If A, B, C be three perfectly elastic balls having masses A, B, C and velocities a, b, c respectively, and A impinges upon B and B upon C, so that their velocities after the impact are u, v and w respectively v being the velocity of B after its impact with C: prove that

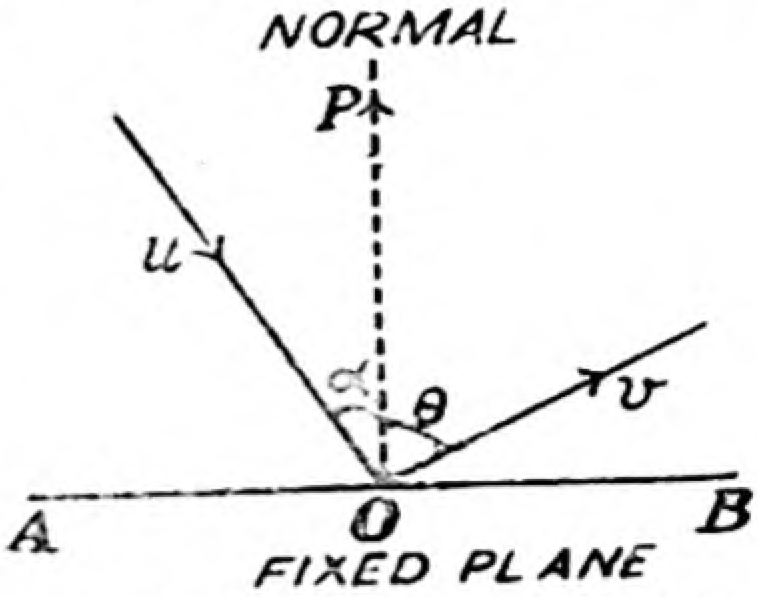
$$A.a^2+B.b^2+C.c^2=Au^2+Bv^2+Cw^2.$$

11. The masses of three perfectly elastic spheres A, B, C are $7m$, $7m$ and m; their centres are in a st. line and C lies between A and B. Initially A and B are at rest and C is given a velocity along the line of centres towards A. Show that it strikes A twice and B once and the final velocities of A, B, C are proportional to 21, 12, 1. *(Delhi)*

9·4 Impact of a smooth sphere on a fixed smooth plane.

A smooth sphere (or particle) of mass m and coefficient of restitution e, impinges obliquely on a fixed smooth plane; to find the subsequent motion. *(Delhi, 1958, 57 ; Panjab, 1957 S)*

Let AB be the fixed plane, O the point at which the sphere impinges and OP the normal to the plane at O. Let the velocity of the sphere at impact be u and the direction of motion of its centre make an angle α with OP. Let the velocity after impact be v making an angle θ with the normal OP.



$\angle\alpha$ is defined as *the angle of incidence*, where $\angle\theta$ is known as **angle of reflection**.

Since the plane and sphere are smooth, there is no force parallel to the plane, hence the component of the velocity of the sphere in this direction remains unaltered.

$$\therefore \quad v\sin\theta = u\sin\alpha \qquad \ldots\text{(I)}$$

Also by Newton's Experimental Law, the relative velocity along the normal after impact is $-e$ times that before impact measured in the same direction.

$\therefore$ If V be the normal component of velocity after impact

$$V = v\cos\theta$$

$$v\cos\theta - 0 = -e(-u\cos\alpha - 0)$$

$$\therefore \quad V = v\cos\theta = eu\cos\alpha \qquad \ldots\text{(II)}$$

The velocity after impact has, therefore, two components $u\sin\alpha$ parallel to AB and $eu\cos\alpha$ parallel to the normal OP. The resultant velocity v after impact is obtained from (I) and (II) by squaring and adding, we have

$$v = u\sqrt{\sin^2\alpha + e^2\cos^2\alpha} \qquad \ldots\text{(III)}$$

and

$$\tan\theta = \frac{u\sin\alpha}{eu\cos\alpha} = \frac{\tan\alpha}{e}$$

or

$$\cot\theta = e\cot\alpha \qquad \ldots\text{(IV)}$$

9·41 More Results.

(1) The impulse of the blow (due to the impact) is measured by the change of momentum along the normal. If the mass of the sphere is m,

$$I = mu\cos\alpha + meu\cos\alpha = mu(1+e)\cos\alpha \qquad \ldots\text{(V)}$$

(2) Loss of K.E. $= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2[1-e^2]\cos^2\alpha \qquad \ldots$(VI)

(3) If $e=1$ the velocity after impact is u and $\theta=\alpha$, the sphere after impact rebounds, *so that the angle of reflection is equal to the angle of incidence*.

(4) If $e=0$, there is no velocity along the normal after impact and the sphere *slides along the plane* with velocity $u\sin\alpha$.

(5) If $\alpha=0$, then $\theta=0$ and $v=eu$

i.e., "If a sphere impinges normally on a plane, it rebounds back with velocity eu".

Ex. 3. *A particle falls from a height h in time t upon a fixed horizontal plane. It rebounds and reaches the maximum height h′ in time t′. Show that $t'=et$ and $h'=e^2h$, where e is the coefficient of restitution. Show also that the whole distance described before the particle has finished rebounding is*

$$\frac{1+e^2}{1-e^2}\,h, \qquad \textit{(Delhi, 1959)}$$

and that the whole time taken is

$$\sqrt{\frac{2h}{g}}\ \frac{(1+e)}{1-e}. \qquad \textit{(Panjab, 1962, 59 ; Delhi, 53)}$$

Let u be the velocity of the particle on first hitting the plane, so that

$$u^2=2gh. \qquad \text{...(I)}$$

The particle rebounds with velocity eu. The velocity when it hits the plane the second time is again eu and the velocity after the second rebound is e^2u. Similarly the velocities after the third, fourth etc., rebounds are e^3u, e^4u etc. The height to which the particle rises after the first rebound is

$$\frac{(eu)^2}{2g}, \qquad \text{...(II)}$$

and after the second $\frac{(e^2u)^2}{2g}$ and so on.

Also $\because$ $u^2=2gh$ so that these distances are e^2h, e^4h, etc.

Thus $$h'=e^2h. \qquad \text{...(III)}$$

The whole distance described is

$$h+2(e^2h+e^4h+\text{......to infinity})$$
$$=h+2h\frac{e^2}{1-e^2}=\frac{h(1+e^2)}{1-e^2}.$$

The time of flight after the first impact (up and down) is $\frac{2eu}{g}$, after the second $\frac{2e^2u}{g}$ etc. and so on and the time of falling originally is $\sqrt{\frac{2h}{g}}$. Thus

$$t'=et \qquad \text{...(V)}$$

Hence the whole time of motion is

$$\sqrt{\frac{2h}{g}}+\frac{2u}{g}\ [e+e^2+e^3\text{......}+\text{to infinity}]$$
$$=\sqrt{\frac{2h}{g}}\left[1+\frac{2e}{1-e}\right]=\sqrt{\frac{2h}{g}}\ \frac{1+e}{1-e}.$$

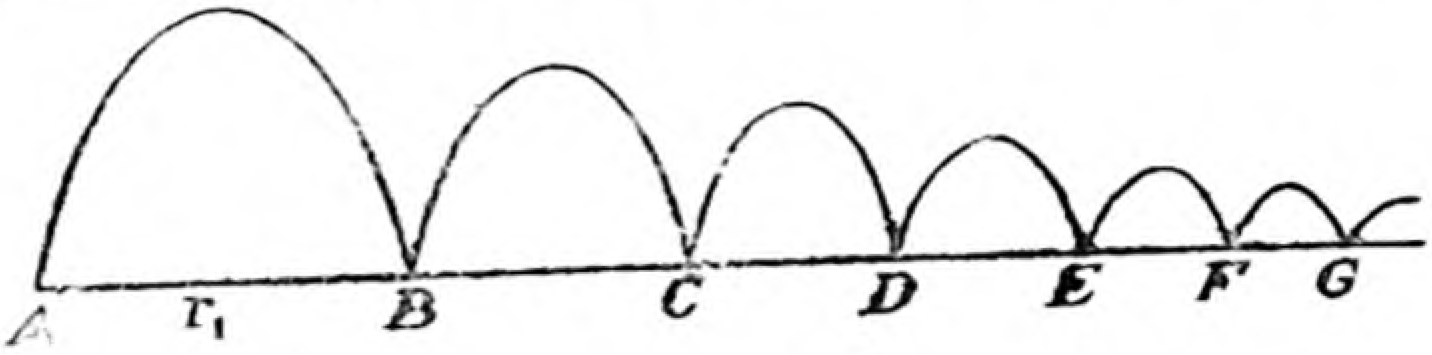
Also note that velocities in the beginning of successive rebounds are as 1 : e.

Ex. 4. *A ball is projected with a velocity u at an angle θ to the horizon, from a point in a smooth horizontal plane. Find the total range and the total time of flight, if e be the coefficient of elasticity.*

The ball moves in a trajectory and the first horizontal range

$$r_1 = \frac{2u^2 \sin\theta \cos\theta}{g}. \qquad \ldots(\text{I})$$

On meeting the plane at B, it rebounds at some angle to the horizontal and again describes a parabola and then again it rebounds at C and so on there is a series of parabolas.



Initial vertical component of velocity $= u \sin\theta$.

Hence the vertical components at the successive rebounds are $eu \sin\theta$, $e^2 u \sin\theta$, $e^3 u \sin\theta$,.........

Time of flight for first parabola $= \dfrac{2u \sin\theta}{g}$

,, ,, ,, ,, second ,, $= \dfrac{2eu \sin\theta}{g}$

,, ,, ,, ,, third ,, $= \dfrac{2e^2 u \sin\theta}{g}$

and so on

$$\therefore \text{ total time of flight} = \frac{2u \sin\theta}{g}\,(1+e+e^2+\ldots\ldots\text{to } \infty) = \frac{2u \sin\theta}{g(1-e)}.$$

(*ii*) Again, $\because$ the plane is smooth, there is no force coming into play along the plane, therefore there is no change of velocity along the horizontal plane, *i.e.*, $u \cos\theta$ remains unaltered.

$\therefore$ total distance covered in time $\dfrac{2u \sin\theta}{g(1-e)}$ in the horizontal direction

$$= \frac{2u \sin\theta}{g(1-e)} \times u \cos\theta$$

$$= \frac{u^2 \sin 2\theta}{g(1-e)}.$$

Afterwards when it ceases to rebound, it moves along the given plane with velocity of $u \cos\theta$.

Exercise IX (B)

1. A ball moving with a velocity of 10 feet per second, impinges on a smooth fixed plane at an angle of 45° ; if the coefficient of restitution be $\frac{4}{5}$, find the velocity and direction of motion of the ball after impact.

2. A sphere of mass 2 lb., moving with a velocity of 12 ft. per second, impinges on a smooth fixed plane in a direction making 60° with the normal to the plane, find velocity and direction of motion after impact, the coefficient of elasticity being 0·75. Find also the loss in kinetic energy and the impulse on the plane due to the impact.

3. A ball falls from a height of 16 feet upon a plane inclined at (1) 30°, (2) 45°, (3) 60° to the horizon ; find the velocity and direction of motion after the impact in the three cases, the coefficient and restitution being $\frac{3}{4}$.

4. A ball is projected vertically with a velocity of 80 ft./sec. The moment it reaches the highest point, a second ball is thrown after it from the same point. How high will they collide ? If the coefficient of restitution be $\frac{3}{5}$, find the time the second ball takes to reach the ground.

5. A ball moving with a velocity of 20 ft./sec. impinges on a fixed smooth plane in a direction making an angle of 30° with the plane, if the coefficient of restitution be $\frac{2}{3}$, find the velocity and direction of motion of the ball after impact.

6. An elastic ball of mass m is projected vertically upwards from a point on a horizontal plane with a velocity u. If the coefficient of restitution be e, find the total space described by it and the time that elapses upon the instant of its nth rebound. What is the kinetic energy after the nth rebound ? *(P.U.)*

7. A steel ball falling from a height of 125 cm. on a horizontal steel plate rises to a height of 51·2 cm. after rebounding twice. What is the coefficient of restitution ?

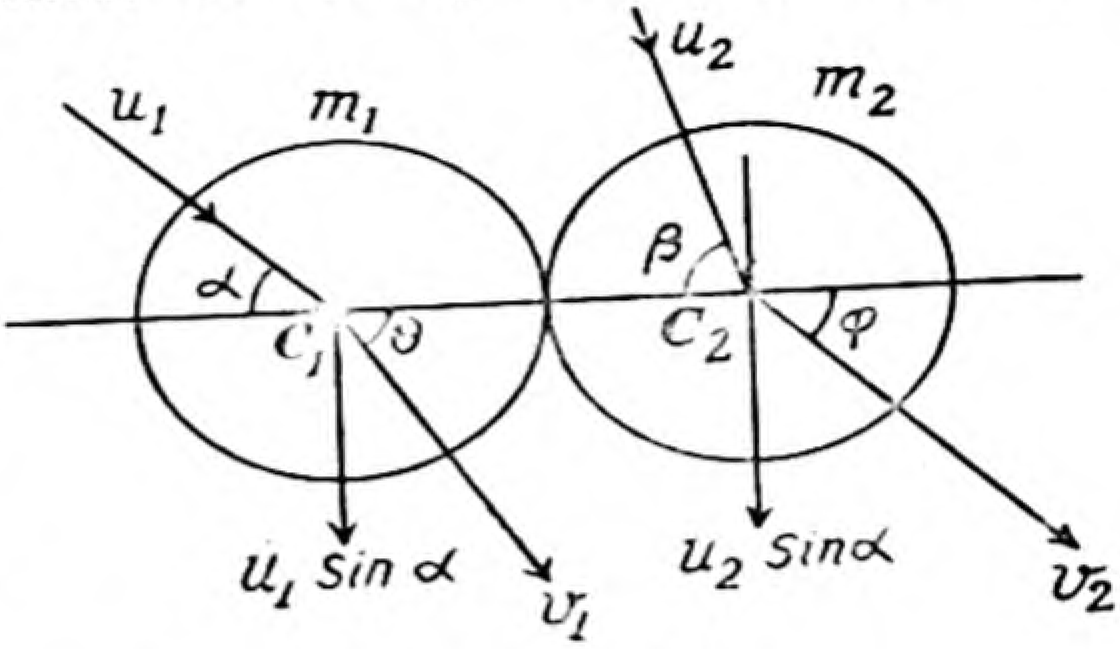
8. A smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Show that a ball whose coefficient of restitution is e, projected along the table from a point on the rim in direction making an angle θ with the radius through the point will return to the point of projection after two impacts if

$$\theta=\tan^{-1}\sqrt{\left(\frac{e^3}{1+e+e^2}\right)}.$$ *(P.U.)*

9. A particle is projected in a direction inclined to the vertical and bounces on the smooth horizontal plane. If the coefficient of restitution be e, and the range of one bound is r find the range of the next. *(P.U.)*

9·5. Oblique impact of two spheres :

(*Delhi, 1961* ; *Panjab, 1961*)



Let C_1, C_2 be the centres of the spheres, m_1, m_2 their masses and let their velocities u_1 and u_2 be inclined at angles α and β to the line of centres C_1C_2 at the moment of impact.

Since there is no force coming into play in the direction perpendicular to the common normal $C_1 C_2$.

$\therefore$ The components of velocity perpendicular to C_1C_2, *i.e.*, $\mu_1 \sin \alpha$ and $\mu_2 \sin \beta$ remain unaltered by the impact.

If v_1, v_2 be the velocities of the two spheres after impact making angles θ and ϕ with the line of centres, we have

$$V_1 \sin \theta = u_1 \sin \alpha \qquad \text{...(I)}$$

$$V_1 \sin \phi = u_2 \sin \beta \qquad \text{...(II)}$$

By the Principle of Conservation of Momentum applied along the common normal, we have

$$m_1 V_1 \cos \theta + m_2 v_2 \cos \phi = m_1 u_1 \cos \alpha + m_2 u_2 \cos \beta \qquad \text{...(III)}$$

By Newton's Experimental law along C_1C_2 :

$$v_1 \cos \theta - v_2 \cos \phi = -e(u_1 \cos \alpha - u_2 \cos \beta) \qquad \text{...(IV)}$$

These four equations determine the four unknown v_1, v_2, θ, and ϕ.

Multiplying (IV) by m_2 and adding to (III), we get

$$v_1 \cos \theta = \frac{u_1 \cos \alpha\ (m_1 - e m_2) + m_2 u_2 \cos \beta\ (1+e)}{m_1 + m_2} \qquad \text{...(V)}$$

Similarly

$$v_2 \cos \phi = \frac{m_1 u_1 \cos \alpha\ (1+e) + u_2 \cos \beta\ (m_2 - e m_1)}{m_1 + m_2} \qquad \text{...(VI)}$$

To obtain v_1 square (I) and (V) and add.

Dividing (I) by (V) we get $\tan \theta$. Similarly (II) and (VI) give v_2 and ϕ.

Cor. 1. The impulse of the blow on m_1 = its change of momentum measured along the common normal

i.e.,
$$m_1(u_1 \cos a - v_1 \cos \theta)$$
$$= \frac{m_1 m_2 (1+e)(u_1 \cos \alpha - u_2 \cos \beta)}{m_1 + m_2} \text{ on reduction.}$$

This is equal and opposite to the impulse on m_2.

Cor. 2. If $u_2=0$, then $v_2 \sin \phi=0$ or $\phi=0$ and $v_2 \neq 0$. In this case the second sphere moves off along the line of centres.

Cor. 3. If
$$m_1=m_2 \text{ and } e=1,$$
$$v_1 \cos \theta=u_2 \cos \beta$$
and
$$v_2 \cos \phi=u_1 \cos \alpha$$
i.e. the spheres interchange their velocities in the direction of the line of centres.

9·51 Loss of Kinetic energy in oblique impact.

The velocities perpendicular to the line of centres are unaltered. The loss of Kinetic energy is, therefore, the same as in the case of direct impact if we substitute $u_1 \cos \alpha$ and $u_2 \cos \beta$ for u_1 and u_2 respectively.

The loss is, therefore,
$$\frac{1}{2} \frac{m_1 m_2}{m_1+m_2}(u_1 \cos \alpha-u_2 \cos \beta)^2(1-e)^2.$$

Ex. 5. *If two equal and perfectly elastic spheres impinge at right angles then directions after impact will still be at right angles.*

(*Panjab B.A./B.Sc., 1959 S*)

If $m_1=m_2$ and $e=1$. Then equations (5) and (6) of §9·5 give
$$v_1 \cos \theta=u_2 \cos \beta \quad \text{and} \quad v_2 \cos \phi=u_1 \cos \alpha$$
Also
$$v_1 \sin \theta=u_1 \sin \alpha \quad \text{and} \quad v_2 \sin \phi=u_2 \sin \beta$$
$$\therefore \quad \tan \theta=\frac{u_1}{u_2} \frac{\sin \alpha}{\cos \beta} \quad \text{and} \quad \tan \phi=\frac{\sin \beta}{\cos \alpha} \frac{u_1}{u_2}$$
$\therefore \quad \tan \theta \tan \phi=\tan \alpha \tan \beta.$
But $\tan \alpha . \tan \beta=-1$,
$\therefore \quad \tan \theta \tan \phi=-1.$

Hence it follows that if the two equal and perfectly elastic spheres impinge at right angles their directions after impact will still be at right angles.

Ex. 6. *A ball overtakes another ball of m times its mass which is moving with* $\frac{1}{n}$ *of its velocity in the same direction. If the impact reduces the first ball to rest, prove that the coefficient of restitution is*
$$\frac{m+n}{mn-m} \text{ and } m>\frac{n}{n-2}.$$

Let u be the velocity and M the mass of the first ball.

Then
$$Mu+mM\frac{u}{n}=M \times 0+mMv$$
and
$$v-0=-e\left(\frac{u}{n}-u\right)$$
$$\therefore \quad v=\frac{n-1}{n} eu \qquad \ldots(1)$$

$$\therefore \qquad u\left(1+\frac{m}{n}\right)=m\left(\frac{n-1}{n}\right)eu$$

or

$$e=\frac{m+n}{mn-m}.$$

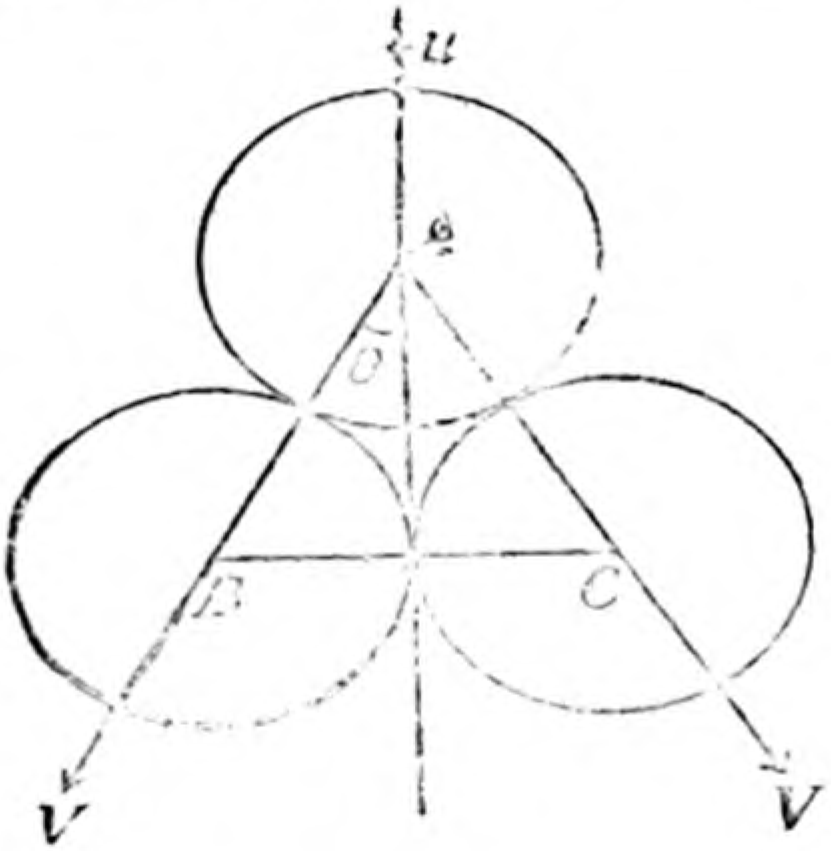
As $\quad e<1,\ m+n<mn-m \text{ or } m>\dfrac{n}{n-2}.$

Ex. 7. *Two equal balls of radius a are in contact and are struck simultaneously by a ball of radius c moving in the direction of their common tangent. If all the balls are of the same material the coefficient of elasticity being e, prove that the impinging ball will be reduced to rest if*

$$2e=\frac{c^2(a+c)^2}{a^3(2a+c)}.$$

(*Agra*)

Let A, B, C be the centres of the three balls. The ball A impinges on the others with a velocity u along the common tangent of B and C and is reduced to rest. B and C move away with the same velocity v after impact along AB and AC because of symmetry.



Apply the Principle of Momentum for all the three balls along the line of motion of A.

$$2mv\cos\theta=m_1u$$

where $\quad m_1=\frac{4}{3}\pi c^3 \quad \text{and} \quad m=\frac{4}{3}\pi a^3$

$$\cos\theta=\frac{\sqrt{(a+c)^2-a^2}}{a+c}$$

$$=\frac{\sqrt{2ac+c^2}}{a+c}.$$

Applying Newton's law along AB or AC

$$v=-e(0-u\cos\theta)=eu\cos\theta$$

$$\therefore \quad 2a^3v\cos\theta=c^3u$$

$$2a^3eu\cos^2\theta=c^3u$$

$$\therefore \quad 2e=\frac{c^3(a+c)^2}{a^3(\sqrt{2ac+c^2})^2}=\frac{c^2(a+c)^2}{a^3(2a+c)}.$$

Exercise IX (C)

1. A ball of mass 5 lbs. and moving with velocity 15 ft. per sec. impinges on a ball of mass 10 lbs. and moving with velocity 5 ft. per sec. If their velocities before impact be parallel and inclined at an angle of 30° to the line joining their centres at the instant of impact find the resulting motion, the coefficient of restitution being $\frac{1}{2}$.

2. A ball of mass 8 lbs. moving with velocity 4 ft. per sec. impinges on a ball of mass 4 lb. moving with velocity 2 ft. per sec. If their velocities before impact be inclined at angles 30° and 60° to the line joining their centres at the moment of impact, find their velocities after impact when $e=\frac{1}{2}$. *(Delhi, 1952)*

3. Two smooth balls one of mass double that of the other move with equal velocities in opposite parallel directions and impinge, their directions of motion at the instant of impact making angles of 30° with the line of centres. If the coefficient of restitution be $\frac{1}{2}$; find the velocities and directions of motion after the impact.

4. Show that if a smooth sphere of mass m impinges obliquely on another smooth sphere of mass m' at rest and if the directions of the spheres after impact are at right angles then $m=em'$.

(P.U. 1965, 68)

5. Two equal billiard balls are in contact on a table, a third equal ball strikes them symmetrically and remains at rest after the impact. Show that the coefficient of restitution is $\frac{2}{3}$.

6. A smooth sphere A impinges on an equal smooth sphere B which is at rest. If the direction of motion of A before and after impact makes angles θ and ϕ with the line of centre of the spheres, prove that

$$\tan\phi=\frac{2\tan\theta}{1-e}$$

and the direction of motion of A is deviated through an angle

$$\tan^{-1}\frac{(1+e)\tan\theta}{(1-e)+2\tan^2\theta}$$

where e is the coefficient of restitution. *(P.U.)*

7. Two equal spheres of radius r, move with the same speed in opposite directions along parallel lines, which are at a distance d apart. Prove that the direction of motion of each deviates, on impact, through a right angle, if

$$d^2(1+\epsilon)=4er^2.$$

8. Two equal balls (elasticity e) impinge, having before impact resolved velocities u_1, v_1 in the direction of the common normal and u_2, v_2 perpendicular to it. If their motions after impact are at right angles, prove that

$$(u_1+v_1)^2+4u_2v_2=e^2(u_1-v_1)^2.$$

9. A smooth sphere approaches an equal stationary sphere, its centre moving along a tangent to the stationary sphere. On impact the path of moving sphere is caused to deviate through 30°. Find the coefficient of restitution and show that they move with equal velocities after impact.

10. A ball impinges on another equal ball moving with the same velocity in a direction perpendicular to its own, the line joining the centres of the balls at the instant of impact being perpendicular to the direction of motion of the second ball, if e be the coefficient of restitution. Show that the direction of motion of the second ball is turned through an angle

$$\tan^{-1}\frac{1+e}{2}.$$

9·6 Compression and Restitution. When a wet tennis ball strikes against a plane, it leaves there a circular patch, not merely a geometrical point.

This shows that at the instant of impact the ball must have undergone a deformation near the point of contact and then recovered its shape. This phenomenon is true of any two impinging elastic bodies in a greater or smaller degree. Thus the period of impact can be divided into two parts, during the first part of which the bodies compress one another and during the second they recover their shapes.

The mutual actions between the two spheres during these two stages are called the *forces of compression* and *forces of restitution.*

(*i*) **The period of compression** is the period during which the distance between the centres of the spheres diminishes until the distance is a minimum at the moment of the greatest compression.

(*ii*) **The period of restitution** is the period during which the spheres resume their original shape and separate completely.

At the instant of greatest compression the distance between the centres of the spheres is minimum and, therefore, their relative rate of change is then zero, *i.e.* the relative velocity of the spheres along their common normal vanishes. Thus at the time of the greatest compression the components of the velocities of the spheres along the line of centres must be the same.

9·61 *To show that the impulse of restitution is e times the impulse of compression.* (*P.U.*)

Let U be the common velocity of the spheres at the instant when compression is over and I, I′ the impulses of the forces of compression and restitution respectively. During the period of compression the loss of momentum in the first sphere is

$m_1(u_1-\text{U})$ and the gain by the second is $m_2(\text{U}-u_2)$.

$$\therefore \qquad \text{I}=m_1(u_1-\text{U})=m_2(\text{U}-u_2). \qquad \ldots(\text{I})$$

Hence
$$\frac{\text{I}}{m_1}+\frac{\text{I}}{m_2}=u_1-u_2$$

Similarly during the period of restitution the loss in momentum of the first sphere is $m_1(\text{U}-v_1)$ and the gain of the second is

$$m_2(v_2-\text{U}).$$

Hence $$I' = m_1(U - v_1) = m_2(v_2 - U)$$

$$\therefore \quad \frac{I'}{m_1} + \frac{I'}{m_2} = v_2 - v_1 \qquad \text{...(II)}$$

$$\therefore \quad \frac{I'}{I} = \frac{v_2 - v_1}{u_1 - u_2} = e.$$

Cor. Total impulse

$$(I + I') = I(1 + e)$$

$$= \frac{m_1 m_2}{m_1 + m_2}(1 + e)(u_1 - u_2) \qquad \text{...(III)}$$

9·62 *Kinetic Energy lost by direct impacts during the two periods.*

U, the common velocity of the spheres at the time of the greatest compression is given by

$$U = \frac{mu + m'u'}{m + m'} = \frac{mv + m'v'}{m + m'}. \qquad \text{...(I)}$$

$\therefore$ E_1 = loss of K.E. during the period of compression

$$= \tfrac{1}{2}mu^2 + \tfrac{1}{2}m'u'^2 - \tfrac{1}{2}(m + m')U^2$$

$$= \tfrac{1}{2}\left[mu^2 + m'u'^2 - \frac{(mu + m'u')^2}{m + m'}\right]$$

$$= \tfrac{1}{2}\frac{mm'}{m + m'}(u - u')^2. \qquad \text{...(II)}$$

E_2 = loss of K.E. during the period of restitution

$$= \tfrac{1}{2}(m + m')U^2 - \tfrac{1}{2}mv^2 - \tfrac{1}{2}m'v'^2$$

$$= -\tfrac{1}{2}\left[mv^2 + m'v'^2 - \frac{(mv + m'v')^2}{m + m'}\right]$$

$$= -\tfrac{1}{2}\frac{mm'}{m + m'}(v - v')^2$$

$$= -\tfrac{1}{2}\frac{mm'}{m + m'}e^2(u - u')^2 = -e^2E_1. \qquad \text{...(III)}$$

$$\therefore \quad E_2 = -e^2E_1. \qquad \text{...(IV)}$$

The negative sign shows that a part of the energy lost in the compression period is regained in the Restitution period.

$\therefore$ total loss in energy

$$= E_1 + E_2 = E_1 - e^2E_1 = (1 - e^2)E_1$$

$$= \tfrac{1}{2}\frac{mm'}{m + m'}(u - u')^2(1 - e)^2.$$

Miscellaneous Exercise VIII

1. Describe the action that takes place when one elastic body impinges on another. Show that the ratio of the impulses of the forces of restitution and compression is equal to the coefficient of restitution. *(P.U.)(Art. 9·61)*

2. When two smooth elastic spheres collide, explain why the components of the velocity of each, perpendicular to the line of centres, is unaltered by the impact.

What will be the direction of motion, after impact, of the sphere A in the following cases? In all cases spheres are smooth and elastic and sphere A, before impact is at rest.

(*i*) Moving sphere B impinges directly on A.

(*ii*) ,, ,, B ,, obliquely on A.

(*iii*) Two equal spheres B, B′ ; moving directly towards A from opposite directions with equal velocities, strike A simultaneously.

(*iv*) Two equal spheres B, B′, moving directly towards A from mutually perpendicular directions with equal velocities, strike A simultaneously. *(P.U. B.Sc., 1960)*

3. If the impulsive force during the period of restitution bears a constant ratio e to the impulsive force during the period of compression, show that

(*i*) the relative velocity after impact is $-e$ times the relative velocity before impact, and

(*ii*) the momentum of the system is the same after impact as before. *(P.U.)*

4. Show that the loss of K.E. in direct impact, when the spheres are moving towards each other, is greater than that in any other case of impact.

5. A smooth sphere impinges obliquely on a smooth fixed plane. If before impact its velocity components along and normal to the plane are u and v, what they will be after impact, e being the coefficient of restitution? *(P.U. B.Sc., 1961 S)*

6. A ball hanging vertically by a string is struck by another equal ball falling vertically downwards, such that at the time of impact its velocity is 10 f.p.s. and the line of centres makes an angle of 30° with the vertical. Find the horizontal and vertical components of the velocity of the falling ball after impact, taking the coefficient of restitution equal to unity. *(P.U. B.Sc., 1960)*

7. A particle is projected with a velocity, whose horizontal and vertical components are u and v respectively, from a point in a horizontal plane, so as to strike a vertical wall at right angles and after rebounding from the wall and once from the horizontal plane, returns to the point of projection. Taking the coefficient of restitution to be e in both cases, find the velocity components after each impact. Show that $e=\frac{1}{2}$.

Also find the total time taken. *(P.U. B.Sc., 1961)*

8. From the foot A of a smooth vertical wall AC, a particle is projected against another smooth parallel vertical wall BD, the plane of projection being perpendicular to both the walls, and after three impacts at B, C, D comes back to the horizontal plane through A. If u, v be the horizontal and vertical components respectively of the velocity of projection, h_1, h_2, h_3 the heights of B, C, D respectively above the horizontal through A, and e the coefficient of restitution, find

(*i*) the horizontal and vertical components of velocity before and after impacts at B, C, D ;

and (*ii*) the greatest height attained by the particle. *(P.U. 1961 S)*

9. An imperfectly elastic sphere of mass m, moving with velocity u, impinges on another sphere of mass m' at rest. The second sphere afterwards strikes a vertical plane at right angles to its path ; show that there will be no further impact of the spheres, if

$$m(1+e'+ee')<e\,m' ;$$

where e, e' are the respective coefficients of restitution between the spheres and the second sphere and the plane.

(Panjab, 1961 S, 69 ; Delhi, M.Sc. ; B.U.)

10. A ball is thrown from a point distant a from a smooth vertical wall against the wall and returns to the point of projection. Prove that the velocity u of projection and the elevation α are connected by the equation

$$u^2 \sin 2\alpha = ag\left(1+\frac{1}{e}\right),$$

where e is the coefficient of restitution between the ball and the wall.

(P.U., 1955 S)

11. In Q. 10 above, if the point of projection is at a maximum distance d from the wall, prove that velocity u is given by,

$$u^2 = gd\left(1+\frac{1}{e}\right).$$

12. A perfectly elastic ball is projected from the foot of a plane inclined at an angle θ to the horizon. If after striking the plane at a distance l from the point of projection it rebounds and retraces its former path, show that the velocity of projection is

$$\sqrt{\frac{gl\,(1+3\sin^2\theta)}{2\sin\theta}}.$$ *(P.U., B.Sc., 1958 & Agra)*

13. An elastic ball is projected from a point A at the foot of one of the two smooth parallel vertical balls so that after three impacts it may return to A. If the last impact be direct, show that

$$e^3+e^2+e=1.$$ *(Rajasthan, 1950)*

14. A smooth elliptical tray is surrounded by a smooth vertical rim. Prove that a perfectly elastic ball projected from a focus along the tray in any direction will after two impacts return to the same focus.

[**Hints :** The normal at any point of an ellipse bisects the interior angle between the focal distances.]

15. A particle is projected from a point on an inclined plane and at the rth impact strikes the plane perpendicularly and at the nth impact is at the point of projection. Prove that

$$e^n - 2e^r + 1 = 0.$$

16. A smooth ring is fixed horizontally on a smooth table and from a point P of the ring a particle is projected along the surface of the table. If e be the coefficient of elasticity between the ring and the particle and α the angle which the direction of projection makes with the radius to P show that the particle will return to the point of projection after two reflections at the ring if

$$\cot^2 \alpha = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3}.$$

17. A particle of elasticity e dropped through a vertical distance of 20 feet, strikes the highest point of a plane which is of length 12 feet and is inclined at 30° to the horizon. If the particle descends to the bottom in three jumps, show that

$$e(1+e)(1+e^2)(1+e+e^2) = 0{\cdot}3.$$

18. A square table ABCD whose side is a has raised edges. A particle of elasticity e is projected from a point P in AB and hits the sides BC, CD, DA in Q, R, S; prove that the paths PQ and RS are parallel.

If α be the angle QPB and BP be x, prove that, if the particle returns to P, $x(1-e) = a(1-e \cot \alpha)$. (*Delhi, 1959 ; Allahabad*)

19. A smooth sphere, of mass m, is tied to a fixed point by an inextensible string and another sphere, of mass m; impinges directly on it with velocity v in a direction making an acute angle α with the string. Show that m begins to move with velocity

$$\frac{m'(1+e) \sin \alpha}{m + m' \sin^2 \alpha}\, v.$$ (*Panjab Hons.*)

20. A smooth sphere after falling from rest through an inextensible string strikes a smooth plane inclined at an angle α to the horizon. Show that the distance between the two points at which it strikes the plane is $he\,(1+e) \sin \alpha$.

Prove also that the whole range on the inclined plane when the particle ceases to rebound is

$$\frac{4\, he \sin \alpha}{(1-e)^2}.$$

ANSWERS

Exercise IX (A), Page 203

1. $e=\frac{1}{2}$. 6. $8\frac{179}{192}$ ft. lbs.

7. (*i*) $\frac{4}{4}$ and $\frac{34}{4}$ resp.

(*ii*) 0 and 4 respectively *i.e.*, they change their velocities.

(*iii*) Their common velocity is $\frac{\mu}{2}$.

8. There will be 3 impacts in order between A and B, then B and C and finally again between A and B. Vel. Ratio $=7:8:12$.

Exercise IX (B), Page 208

1. $v=\sqrt{82}$ f.p.s. ; $\theta=\tan^{-1}\frac{5}{4}$ to the normal.

2. $v=\dfrac{3\sqrt{57}}{2}$ f.p.s. ; $\theta=\tan^{-1}\dfrac{4}{\sqrt{3}}$ to the normal.

Loss in K.E. $=15\frac{3}{4}$ ft. lbs.
Impulse $=21$ F.P.S. units.

3. $v=4\sqrt{43}$, $\theta=\tan^{-1}\dfrac{4}{3\sqrt{3}}$.

4. They meet 5 ft. above the ground. Vel. of second ball after impact

$$=24 \text{ f.p.s.} \; ; \quad t=\frac{2\sqrt{21}-3}{4} \text{ sec.}$$

5. $v=2\sqrt{79}$ f.p.s. ; $\theta=\tan^{-1}\dfrac{5\sqrt{3}}{2}$.

6. Total space $=\dfrac{u^2}{g}\left(\dfrac{1-e^{2n}}{1-e}\right)$; Time $=\dfrac{24}{9}\left(\dfrac{1-e^n}{1-e}\right)\left(\dfrac{1-e^n}{1-e}\right)$.

Vel. after nth rebound $=e^n.u$.
K.E. ,, ,, ,, $=\frac{1}{2}mu^2e^{2n}$.

7. $e=\frac{4}{5}$. 9. Next range $=er$.

Exercise IX (C), Page 211

1. $v=5\sqrt{3}$ ft.p.s. at 60° to the line of centres.

2. $v_1=\dfrac{\sqrt{29+4\sqrt{3}}}{2}$; $v_2=\sqrt{15}$ f.p.s. ; $\tan\theta=\dfrac{5}{2\sqrt{3}+1}$.

3. $\theta=90°$, $v=\frac{4}{2}$ $\phi=30°$ $v'=u$.

9, $e=\frac{1}{3}$.

Miscellaneous Exercise VIII, Page 214

1. See art. 9·6, 9·61.

2. (*i*) and (*ii*) along the line of centres.
(*iii*) remains at rest
(*iv*) 45° with the vertical.

5. u and ev respectively.

6. $\frac{5\sqrt{3}}{2}$ f.p.s. and $\frac{5}{2}$ f.p.s. respectively.

7. The horizontal and vertical components after the first impact are eu and zero respectively and after the second impact they are eu and ev. Total time $=\frac{2v}{g}(1+e)$. Also $e=\frac{1}{2}$.

8. The components after the impacts at B, C, D are respectively

eu, $\sqrt{v^2-2gh_1}$; e^2u, $\sqrt{v^2-2gh_2}$ and ϵ^3u, $\sqrt{v^2-2gh_3}$; $\frac{v^2}{2g}$.

CHAPTER X

RELATIVE MOTION

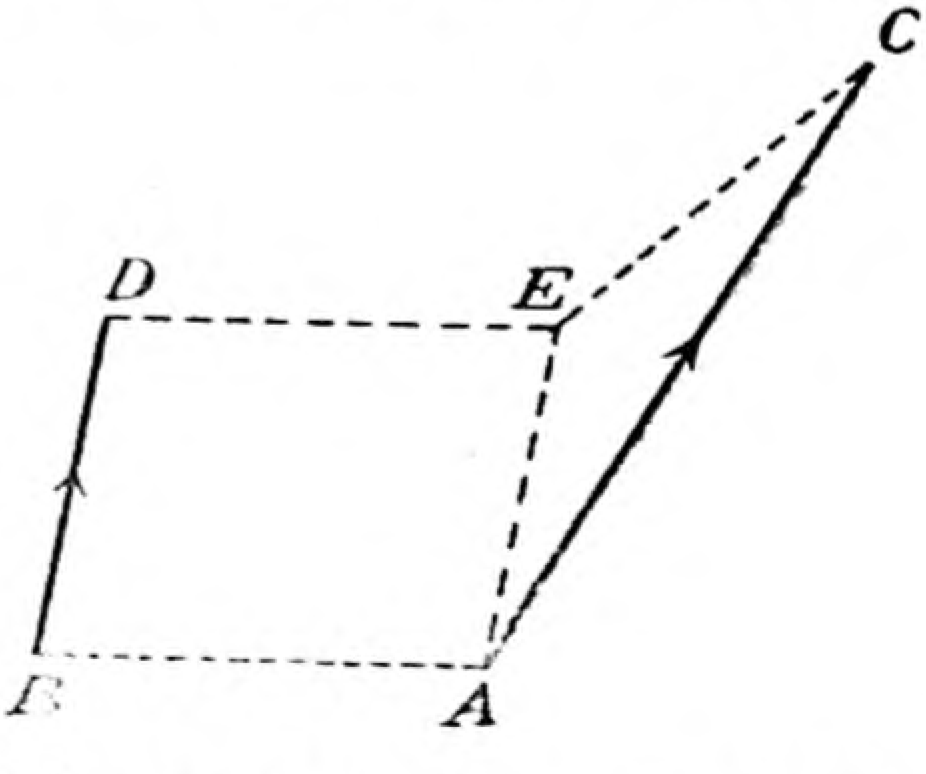
10·1 Relative Motion. We know that "All rest or motion is relative"—that is, with respect to the surrounding objects. Thus if an object does not change its position with respect to its surroundings, we say it is at rest; though the very earth itself on which the problem of rest or motion is being considered, is moving ceaselessly in the universe.

10·11 Def. Relative Motion. When the distance between two points is changing either in magnitude or in direction or in both (magnitude and direction), then either is said to have a *motion relative to the other* or *a velocity relative to the other.*

10·12 Def. The velocity of a point A relative to another point B, is the velocity which A appears to have when viewed from B.

10·2 To show that *the velocity of A relative to B is obtained by compounding the absolute velocity of A, with the reversed velocity of B.*
(*P.U. B.Sc. 1959 S*)

Proof. Let the vectors $\overline{\text{AC}}$ and $\overline{\text{BD}}$ represent the velocities of A and B respectively, moving in a plane. Join AB and complete the parallelogram ABCD. Join EC.



By triangle of velocities,

$$\text{AC} = \overline{\text{AE}} + \overline{\text{EC}}$$

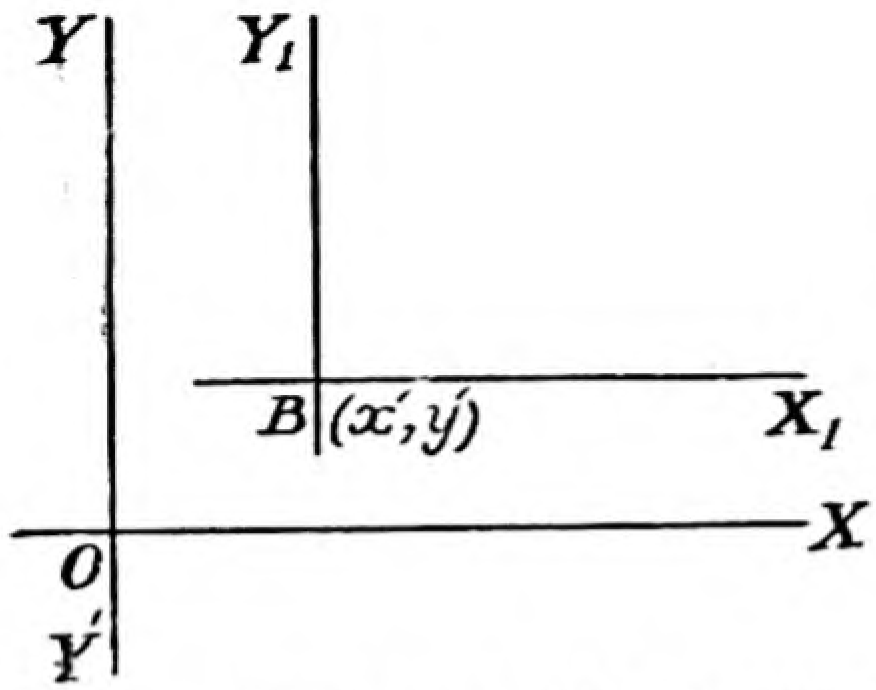
$$= \overline{\text{BD}} + \overline{\text{EC}}$$

Now EC represents the velocity of A viewed from B.

∴ velocity of A is equivalent to velocity of B together with the velocity of A relative to B.

Hence, the velocity A relative to B is obtained by compounding with the actual velocity of A, a velocity equal and opposite to the actual velocity of B.

10·21 Analytical Proof. Let XOX′, YOY′ be two perpendicular axes and A(x, y), B(x', y') be the positions of the two particles at an instant t. Let (X, Y) be the co-ordinates of A referred to the set of parallel axes through B.



Then $x = x' + \mathrm{X}$ ...(I)

$y = y' + \mathrm{Y}$ ...(II)

Transposing and differentiating *w.r.t.* time t, we get

$$\frac{d\mathrm{H}}{dt} = \frac{dx}{dt} - \frac{dx'}{dt} \quad \text{...(III)}$$

$$\frac{d\mathrm{Y}}{dt} = \frac{dy}{dt} - \frac{dy'}{dt} \quad \text{...(IV)}$$

(III) and (IV) show that when all velocities are measured relative to a given point (O here) the velocity component (in any direction) of the moving point A, relative to another moving point B, is obtained by compounding with the actual velocity component of A, the reversed velocity component of B.

Thus, the velocity of A, relative to B, is the resultant of

(*i*) the actual velocity of A, and

(*ii*) the reversed velocity of B. $\therefore$ **Q E.D.**

Cor. 1. *When A and B are moving in the same st. line,* with velocities v_1 and v_2 ; the velocity of A relative to $\mathrm{B} = v_1 - v_2$.

Cor. 2. The velocity of A is equivalent to the velocity of B, compounded with the velocity of A relative to B.

The result follows from the equations I and II of Art. 10·21.

Differentiating *w.r.t.* time t, we get,

$$\frac{dx}{dt} = \frac{dx'}{dt} + \frac{d\mathrm{X}}{dt}$$

$$\frac{dy}{dt} = \frac{dy'}{dt} + \frac{d\mathrm{Y}}{dt} \qquad \therefore \ \text{Q.E.D.}$$

10 3 *Expression for the magnitude and direction of the velocity of A relative to B* (when A and B are moving in a plane).

Let the velocities v_1 and v_2 of A and B make angles α_1 and α_2 with the fixed line XOX′ at any instant t and let V, the velocity of A relative to B, make an angle θ with XOX′.

Ehen by *Art. 10·21* (III) and (IV), we have :

$$\mathrm{V} \cos \theta = v_1 \cos \alpha_1 - v_2 \cos \alpha_2 \quad \text{...(I)}$$

$$\mathrm{V} \sin \theta = v_1 \sin \alpha_1 - v_2 \sin \alpha_2 \quad \text{...(II)}$$

Whence, on squaring and adding

$$\mathrm{V}^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos(\alpha_1 - \alpha_2) \quad \text{...(III)}$$

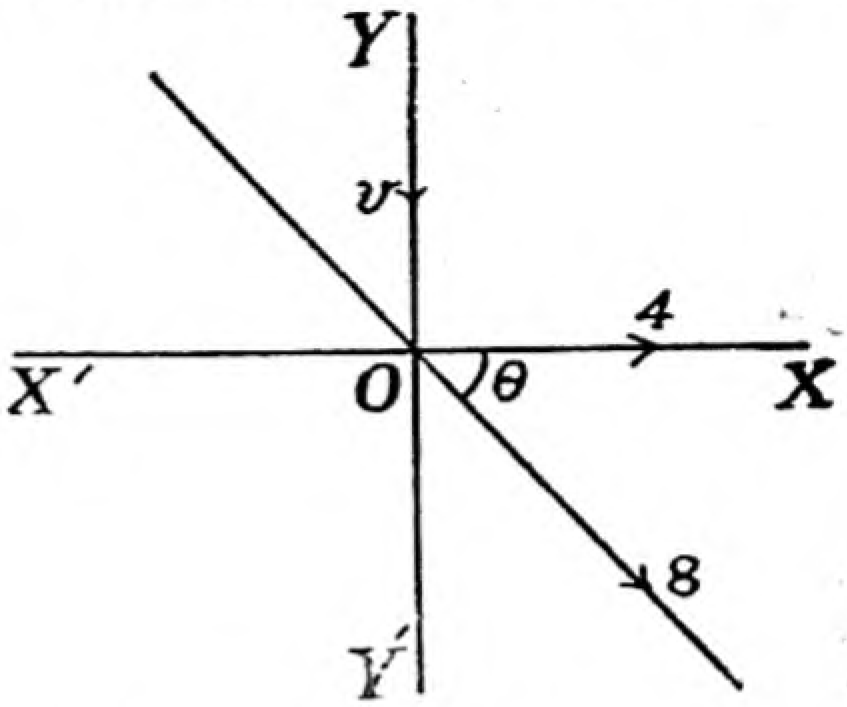
and on dividing

$$\tan \theta = \frac{v_1 \sin \alpha_1 - v_2 \sin \alpha_2}{v_1 \cos \alpha_1 - v_2 \cos \alpha_2}. \quad \text{...(IV)}$$

SOLVED EXAMPLES

Ex. 1. *To a man walking at the rate of 4 m.p.h., rain appears to fall vertically. If its real velocity is 8 m.p.h., find its real direction.* **(*Panjab, 1948*)**

Let OX be the direction of motion of the man and YO the apparent direction of motion of the rain.



By *Art. 10 21* (*Cor. 2*), Real velocity of the rain is equivalent to the real velocity of man compounded with the velocity of rain relative to the man.

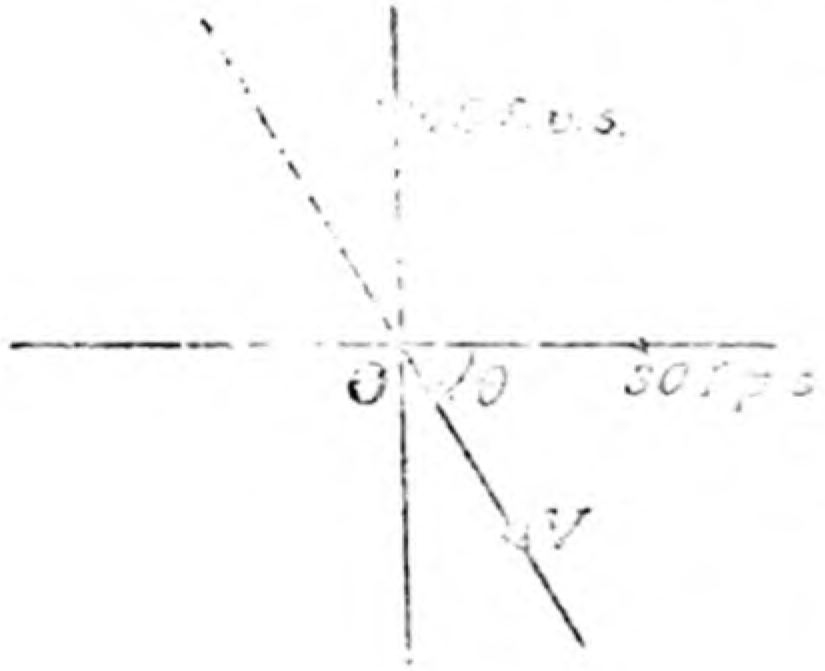
If the real velocity of rain makes $\angle\theta$ with OX, then

$8\cos\theta = 4$. (resolving along OX)

$\therefore\quad \theta = 60°$.

Ex. 2. *A road and a railway line cross at right angles. If at a certain instant a car is approaching the crossing at 80 f.p.s. and a train at 40 f.p.s., find the magnitude and direction of the velocity of the rain relative to the car at the said instant.* **(*Panjab, 1951*)**

Let the required velocity (V) make an angle θ with the direction of motion of car (XOX′) then resolving along OX and OY. **(*Art. 10·21*)**



$$V\cos\theta = 0-(-30)$$
$$= +30$$
$$V\sin\theta = -40-0$$
$$= -40$$

whence

$$V = \sqrt{30^2+40^2}$$
$$= 50 \text{ f.p.s.}$$
$$\tan\theta = -\tfrac{4}{3}.$$

Exercise X (A)

1. Explain and obtain the relative velocity of a moving point P *w.r.t.* another moving point Q.

Two particles are started simultaneously from points A and B which are 5 ft. apart, one from A towards B with a velocity which would cause it to reach B in 3 seconds, and the other at right angles to the direction of the former with $\frac{3}{4}$ of its velocity. Find their relative velocity in magnitude and direction, the shortest distance between them and the time when they are nearest. **(*P.U.*)**

2. A train whose mass is 112 tons is travelling with a uniform speed on a level track and the air resistance etc. is 16 lbs. per ton.

Part of the train, of mass 12 tons, becomes detached. Assuming that the force exerted by the engine is the same throughout, and confining to the time during which the detached part remains in motion, find the acceleration and velocity at time t, of the train relative to the detatched part. Also find how much the train would have gained on the part at time t ? (*Panjab, B.Sc., 1961*)

3. Two motor cars are moving uniformly along two straight roads making an angle of 60°, with each other with velocities of 20 m.p.h. If the first car be moving towards and the second away from the junction of the roads, find the relative velocity of the first *w.r.t.* the second. (*Panjab, 1949 S*)

4. A person, travelling eastward at the rate of 4 m.p.h. finds that the wind seems to blow directly from the North. On doubling his speed, it appears to come from the North-East. Find the direction of the wind and the velocity. (*Panjab, 1950*)

5. A river is $\frac{1}{4}$ mile wide and flows at the rate of $1\frac{1}{2}$ m.p.h. A man who can swim at 2 m.p.h. in still water wishes to cross directly to a point 110 yards downstream. Prove that the time of crossing is 8·9 minutes. (*Panjab, 1951 S*)

6. Two cars A and B are moving uniformly on two straight perpendicular roads at 40 and 20 m.p.h. respectively. A passes the intersection of the roads when B has still to more 50 miles to reach it. How much time will elapse before they are close together and what is the shortest distance between them ?

7. A bomber moves due east at 100 m.p.h. over a town x at a certain time. Six minutes later a pursuit plane sets off from a station y, which is 40 miles due south of x, and flies north-east. If both maintain their courses, find the velocity with which the plane must fly in order to overtake the bomber. (*Panjab, 1953 S*)

8. If the pursuit plane flies at the rate of $150\sqrt{2}$ miles per hour along its course, find its nearest approach to the bomber in miles. (*P.S.C.*)

9. (*a*) Two particles A and B move in a plane with velocities u and v inclined at angles α and β to the axis of x. Find the velocity of B relative to A.

(*b*) Light signals travel with velocity c and earth moves in space with velocity v. A light signal is sent from a point O in time t. Another light signal is sent from the point O to another mirror at a point B and returns after reflection to the point O in the same time t. If $OA=d_1$, $OB=d_2$, and OA and OB are respectively in the direction and perpendicular to the direction of motion of the earth, show that

$$d_1=d_2\sqrt{1-\frac{v^2}{c^2}}.$$

(*Panjab, 1960*)

10. Two straight roads OX and OY are inclined to each other at an acute angle α. One car moves along XO with constant speed

u while a second car moves along OY with constant speed v. If the first car is at distance d from O, when the second is at O, show that the cars are at their least distance apart after time

$$\frac{d(u+v\cos\alpha)}{u^2+v^2+2uv\cos\alpha}$$

and at that time the ratio of their distance from O is

$$\frac{v+u\cos\alpha}{u+v\cos\alpha}.$$

(Panjab, B.Sc., 1959 S)

11. A wheel rolls without slipping along a horizontal plane in a st. line. If n be the velocity of its centre and AB is any diameter, show that the velocity of A relative to B is $2u$ in a direction perpendicular to AB.

12. Two particles start simultaneously from the same point and move along two straight lines, one with uniform velocity and the other from rest with uniform acceleration f. Show that the least relative velocity is $u \sin\alpha$ and that it occurs after a time $\frac{u\cos\alpha}{f}$ where α is the angle between the two lines.

Also find the equation of their relative path.

(P.U., 1968 S ; 1969)

10·4 Relative Acceleration.

From equations (III) and (IV) of Art. 10·21

$$\frac{dX}{dt}=\frac{dx}{dt}-\frac{dx'}{dt} \quad \text{...(III)}$$

$$\frac{dY}{dt}=\frac{dy}{dt}-\frac{dy'}{dt} \quad \text{...(IV)}$$

Differentiating them *w.r.t.* time t again, we have

$$\frac{d^2X}{dt}=\frac{d^2x}{dt^2}-\frac{d^2x'}{dt^2} \quad \text{...(V)}$$

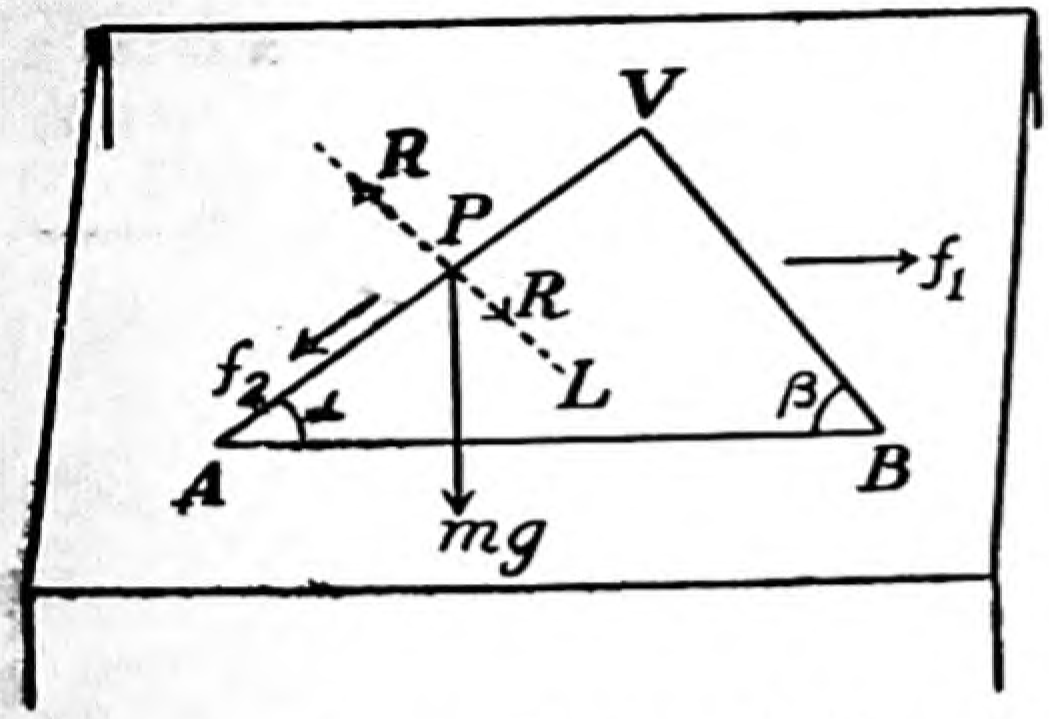
$$\frac{d^2Y}{dt^2}=\frac{d^2y}{dt^2}-\frac{d^2y'}{dt} \quad \text{...(VI)}$$

which show that the component Relative Acceleration of A *w.r.t.* B, in any direction, is obtained by compounding with the component of the acceleration of A, the component of the reversed acceleration of B.

10·5 Motion of a particle on a moving Wedge.

A particle of mass m slides from rest down the smooth face of inclination α, of a wedge. The wedge is of mass M and is free to move on a smooth horizontal table. To discuss the motion.

Let AVB be the wedge (mass M) with vertex V, moving with an acceleration f_1 in the horizontal direction.



Let the particle (mass m) slide down VA, with acceleration f_2.

The forces acting on the wedge are :

(*i*) weight Mg, vertically downwards.

(*ii*) R, the pressure of the mass on the wedge ⊥ to face VA.

(*iii*) R′ the pressure of the table on the wedge vertically upwards (⊥ to face AB).

The forces on the mass m are :

(*i*) weight mg, vertically downwards.

(*ii*) R, the pressure of the wedge on m.

The acceleration of the mass is the resultant of the acceleration of the wedge and the acceleration of the particle relative to the wedge.

∴ Resolving along and perpendicular (PL) to the face VA, we have the accelerations

(*i*) $f_2 - f_1 \cos\alpha$

(*ii*) $f_1 \sin\alpha$ (*along PL*)

Hence the respective equations of motion are :

$$m(f_2 - f_1\cos\alpha) = mg\sin\alpha \qquad \ldots(\text{I})$$

$$mf_1\sin\alpha = mg\cos\alpha - \text{R} \qquad \ldots(\text{II})$$

For the wedge,

$$\text{M}f_1 = \text{R}\sin\alpha \qquad \ldots(\text{III})$$

Whence on solving, we have

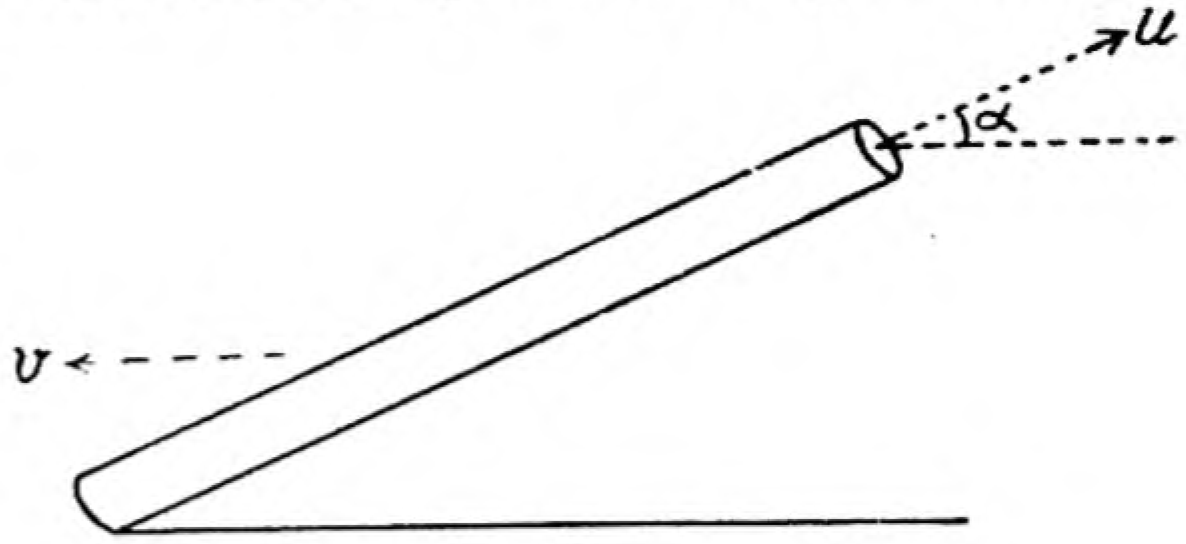
$$f_1 = \frac{mg\sin\alpha\cos\alpha}{\text{M} + m\sin^2\alpha} \;;\; f_2 = \frac{(\text{M}+m)g\sin\alpha}{\text{M}+m\sin^2\alpha}$$

and $$\text{R} = \frac{\text{M}mg\cos\alpha}{\text{M}+m\sin^2\alpha}.$$

Also $$\text{S} = \text{M}g + \text{R}\cos\alpha = \frac{\text{M}(\text{M}+m)\,g}{\text{M}+m\sin^2\alpha}.$$

10·6 Motion of a shot from a gun free to recoil.

A shot of mass m is fired, with a velocity u at an angle α from a gun of mass M free to recoil horizontally with a velocity v, to discuss the motion.



By Cor. 2 of *Art. 10·21*, the horizontal component of actual velocity V of the shot = horizontal velocity of the shot relative to the gun compounded with the velocity of the gun

$$=u\cos\alpha-v$$

or

$$V\cos\theta=u\cos\alpha-v \qquad \ldots(\mathrm{I})$$

Also the vertical component of the velocity of the shot

$$=u\sin\alpha-0=u\sin\alpha$$

$$V\sin\theta=u\sin\alpha \qquad \ldots(\mathrm{II})$$

If θ be the inclination of the actual velocity of the shot to the horizontal, then

$$\tan\theta=\frac{u\sin\alpha}{u\cos\alpha-v} \qquad \ldots(\mathrm{III})$$

Again, because at the instant of firing, there is no external force, the Principle of Conservation of Momentum may be applied in the horizontal direction,

whence $$m(u\cos\alpha-v)-Mv=0$$

giving $$v=\frac{mu\cos\alpha}{M+m}.$$

$$\therefore \qquad \tan\theta=\frac{M+m}{Mu\cos\alpha}\cdot u\sin\alpha=\left(1+\frac{m}{M}\right)\tan\alpha \qquad \ldots(\mathrm{IV})$$

From (I) and (II), by squaring and adding

$$V^2=u^2+v^2-2uv\cos\alpha \qquad \ldots(\mathrm{V})$$

(IV) and (V) give us the required velocity of the shot.

Ex. 3. *Two straight railways converge to a level crossing at an angle α and two trains are moving towards the crossing with velocities u and v. If a and b are the initial distances of the trains from the crossing, show that their least distance apart will be*

$$\frac{(bu-av)\sin\alpha}{\sqrt{u^2+v^2-2uv\cos\alpha}}.$$

(P.U. 1962)

Let A and B be the original positions of the trains.

Then after time t, if they are at P and Q.

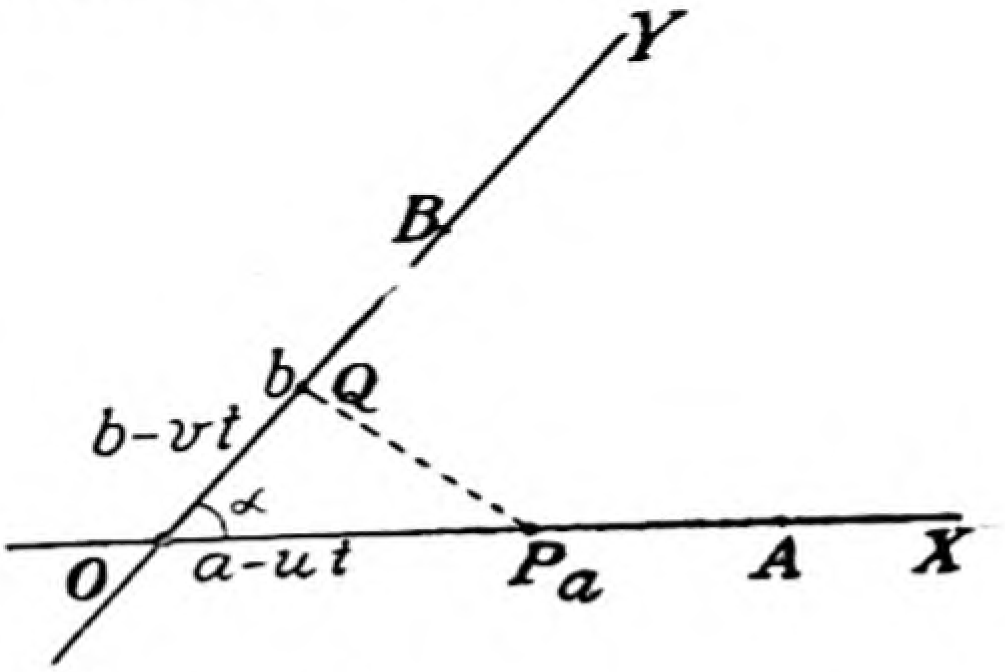
$OP = a-ut$, $OQ = b-vt$

$$y^2 = PQ^2 = OP^2 + OQ^2 - 2OP.OQ.\cos\alpha$$

$$= (a-ut)^2 + (b-vt)^2 - 2(a-ut)(b-vt)\cos\alpha$$

y will be least, when y^2 is least.

Let $z = y^2 = (a-ut)^2 + (b-vt)^2 - 2(a-ut)(b-vt)\cos\alpha \quad \text{...(I)}$



For minima of z, $\dfrac{dz}{dt} = 0 \quad (i)$

and $\dfrac{d^2z}{dt^2} > 0 \quad (ii)$

$$\frac{dz}{dt} = 2(a-ut)(-u) + 2(b-vt)(-v) - 2\cos\alpha(2uvt - av - bu) \quad \text{...(II)}$$

Using (i) ; $t = \dfrac{(a-b\cos\alpha).u + (b-a\cos\alpha).v}{u^2+v^2-2uv\cos\alpha} \quad \text{...(III)}$

Also $\dfrac{d^2z}{dt^2} = u^2+v^2-2uv\cos\alpha$

$$= (u-v)^2 + 2uv(1-\cos\alpha) = (u-v)^2 + 4uv > 0$$

Hence, z is minimum for the above value of t.

(ii) For this value of t, the value of z or PQ^2 is to be found.

From (II),

$$u(a-ut) + v(b-vt) + u\cos\alpha\,(vt-b) + v\cos\alpha\,(ut-a) = 0$$

or $(a-ut)(u-v\cos\alpha) + (b-vt)(v-u\cos\alpha) = 0.$

$$\therefore \quad b-vt = \frac{-(u-v\cos\alpha)}{v-u\cos\alpha}(a-ut) \quad \text{...(IV)}$$

Now $z = (a-ut)[(a-ut)-(b-vt)\cos\alpha] + (b-vt)[(b-vt)-(a-ut).\cos\alpha]$

$$= (a-ut)\left[(a-ut) + \frac{(u-v\cos\alpha)(a-ut)}{v-u\cos\alpha}.\cos\alpha)\right] \text{ [using(IV)]}$$

$$+ \frac{(u-v\cos\alpha)(a-ut)}{v-u\cos\alpha}\left[\frac{(u-v\cos\alpha)(a-ut)}{v-u\cos\alpha} - (a-ut)\cos\alpha\right]$$

$$=(a-ut)^2\left[1+\frac{u\cos\alpha-v\cos^2\alpha}{v-u\cos\alpha}+\frac{u-v\cos\alpha}{v-u\cos\alpha}\left(\frac{u-v\cos\alpha}{v-u\cos\alpha}+\cos\alpha\right)\right]$$

$$=(a-ut)^2\ \frac{(u^2+v^2-2uv\cos\alpha)\sin^2\alpha}{(v-u\cos\alpha)^2} \qquad ...(V)$$

Also $\quad a-ut=a-u\left[\dfrac{(a-b\cos\alpha)u+(b-a\cos\alpha)v}{u^2+v^2-2uv\cos\alpha}\right]$

$$=\frac{(u\cos\alpha-v)(bu-av)}{u^2+v^2-2uv\cos\alpha} \qquad ...(VI)$$

Whence from (V) and (VI)

$$z=\frac{(bu-av)^2\sin^2\alpha}{u^2+v^2-2uv\cos\alpha}.$$

$$\therefore\quad PQ=\sqrt{z}=\frac{(bu-av)\sin\alpha}{u^2+v^2-2uv\cos\alpha}.$$

10·7 Relative Angular Velocity.

To find the angular velocity of a moving point P relative to another moving point Q.

In chapter VIII, we proved that if Q is a fixed point, then the angular velocity of P, relative to Q

$$=\frac{\text{Vel. of P} \perp \text{QP}}{\text{QP}}$$

so that the component of the velocity of P along QP does not have any effect on the direction of PQ.

Now consider the case when Q is also moving. Let u, v be the velocities of P and Q and let them make angles α and β with QP, at any instant t.

Then, the angular velocity of P relative to Q (or the angular velocity of the line PQ)

$$=\frac{\text{component of vel. of P relative to Q in the direction } \perp \text{QP}}{\text{QP}}$$

$$=\frac{u\sin\alpha-v\sin\beta}{\text{QP}}.$$

It may be noted that components of the velocities of P and Q along QP have no effect on the direction of PQ.

Cor. 1. When $u \sin \alpha = v \sin \beta$, P has no velocity relative to Q, *i.e.* the line PQ moves parallel to itself and the linear velocity of P relative to Q is along PQ.

Cor. 2. If PQ is of constant length a; then the angular velocity of P relative to Q

$$= \frac{u \sin (\alpha - \beta)}{a \cos \beta}.$$

Proof. As PQ is constant; velocity components of P and Q along PQ are equal, *i.e.* $u \cos \alpha = v \cos \beta$.

$\therefore$ Reqd. Relative Angular Velocity $= \dfrac{u \sin \alpha - v \sin \beta}{a}$

$$= \frac{u \sin \alpha - \dfrac{u \cos \alpha}{\cos \beta} . \sin \beta}{a}$$

$$= \frac{u \sin (\alpha - \beta)}{a \cos \beta}.$$

Miscellaneous Exercise IX

1. If the particles P and Q are travelling in straight lines with uniform velocities u and v towards a point O, show that the least distance between them during the motion is $\dfrac{u - v}{v} p$; where p is the perpendicular distance of O from the direction of the velocity of P relative to Q, when Q and P are equidistant from O. *(P.U.)*

2. If two projectiles are moving under gravity alone, show that relative to one, the other appears to move uniformly in a straight line.

3. To a person travelling due east, the wind appears to come from the north east but when he doubles his speed, it appears to come from a direction $\tan^{-1} \frac{2}{3}$, north of east. Find the direction of the wind. *(P.U.)*

4. Two scouts, cycling at 9 and 12 m.p.h. along straight roads, which cross at right angles at O, are at some instant both at a distance of one mile from O. Find the shortest distance between the scouts in the subsequent motion. *(P.U.)*

5. ABC is a triangle right angled at C. A particle P starts from A and moves along AC with uniform velocity u; a second particle Q starts from C at the same instant and moves along CB with a uniform velocity v ; show that the shortest distance between them will be $v.AC\sqrt{u^2+v^2}$; after a time

$$\frac{u.AC}{\sqrt{u^2+v^2}}.$$

6. A particle of mass 2 lbs. is placed on a smooth face of an inclined plane of mass 7 lbs. and slope 30°, which is free to slide on a smooth horizontal plane. Show that if the system starts from rest, the particle will slide down a distance of 15 ft. along the plane in 1·25 seconds. *(P.U.)*

7. A body of mass m with two plane smooth faces inclined at an acute angle α is placed with one face in contact with a smooth horizontal table, and a particle of mass $2m$ slides down the other face under gravity. Show that the horizontal acceleration of the body and the pressure on the table are respectively

$$\frac{g \sin 2\alpha}{2-\cos 2\alpha}; \quad \frac{3mg}{2-\cos 2\alpha}.$$

8. Two particles A and B are moving along two concentric circles of radii a and b in a plane, with velocities u and v. Show that when $\angle POQ=\alpha$, the velocity of A relative to B, is

$$\sqrt{u^2+v^2-2uv \cos \alpha}\,;$$

and that when the relative angular velocity is zero,

$$\alpha=\cos^{-1}\frac{au+bv}{av+bu}.$$

9. In Q. 8 above, if the velocities of the particles vary inversely as the radii of the circles ; show that the relative velocity is parallel to the line AB, when

$$\alpha=\cos^{-1}\frac{2ab}{a^2+b^2}.$$

10. In Q. 8 above, if the velocities of A and B vary inversely as the square root of the radii ; show that their relative angular velocity vanishes when

$$\alpha=\cos^{-1}\frac{\sqrt{ab}}{a+b-\sqrt{ab}}.$$

ANSWERS

Exercise X (A), Page 221

1. Rel. Vel. $=\frac{25}{12}$ f.p.s. ; tan $\theta=\frac{3}{4}$. Least distance $=3$ ft.
Time $=1\frac{23}{25}$ sec.

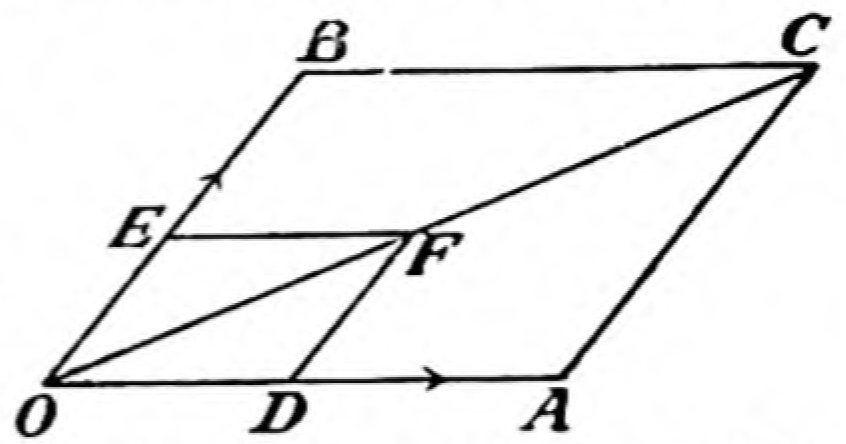
2. $\frac{32}{125}$, Rel. Vel. $=\frac{32}{125}t$.
Relative distance $=\frac{16t^2}{125}$.

3. $20\sqrt{3}$ m.p.h.
4. $x=4\sqrt{2}$ m.p.h. from North-West.
6. Cars are nearest after 30 minutes. Least distance$=20\sqrt{5}$ miles.
7. $\dfrac{400\sqrt{2}}{3}$ m.p.h. **8.** Shortest distance$=\sqrt{10}$ miles.

Miscellaneous Exercise IX, Page 228

1. From a direction at angle of $\tan^{-1}\frac{1}{2}$ of West of North.
4. $\frac{1}{8}$ mile before they reach O.

APPENDIX



1. Law of Parallelogram of Velocities : If a moving point possesses two velocities represented in magnitude and direction by the two sides of a parallelogram drawn from a point, their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through the point.

Let the two velocities, *u* and *v*, be represented in magnitude and direction by the lines OA and OB respectively. Complete the parallelogram OACB.

We may imagine the point to move along OA with velocity *u* and the line OA moves parallel to itself so that the end O describes the line OB with velocity *v*.

In unit time the point moves the distance OA while the line OA moves to the position BC. Thus the moving point is at C after unit time.

At any intermediate time *t*, the displacement of the point will be *ut* along OA and the line OA will move through a distance *vt* parallel to itself. Let these distances be represented by the length OD and OE respectively. Let OF be the diagonal of the parallelogram of which OD and OE are the sides. Join FC.

Then in triangles ODF and OAC,

$$\frac{DF}{OD}=\frac{OE}{OD}=\frac{vt}{ut}=\frac{v}{u}=\frac{OB}{OA}=\frac{AC}{OA}.$$

∴ O, F and C lie on the same straight line.

∴ *Even at any intermediary instant*, the particle moves along OC which, therefore, is the direction of the resultant velocity.

Let V be the resultant velocity so that

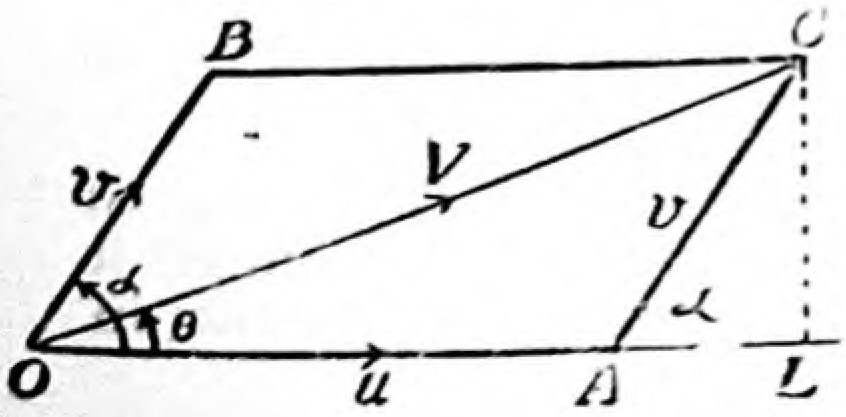
$$OF=Vt$$

$$\because \quad \frac{OF}{OC}=\frac{OA}{OD}=\frac{ut}{u}=t.$$

$$\therefore \quad OC=\frac{OF}{t}=\frac{Vt}{t}=V.$$

Hence OC represents both magnitude and direction of the resultant of velocities *u* and *v* represented by OA and OB.

2. To find the Resultant of two given Velocities u and v :



Let *u* and *v* act along OA and OB and let $\angle AOB=\alpha$.

Then by parallelogram of velocities, the diagonal OC represents the resultant of $\|^{m}$ OACB.

Now $$OC^2=OA^2+AC^2-2OA.AC\cos\angle OAC$$
$$=OA^2+AC^2-2OA.AC\cos(180^\circ-\alpha)$$
$$=OA^2+AC^2+2OA.AC\cos\alpha$$

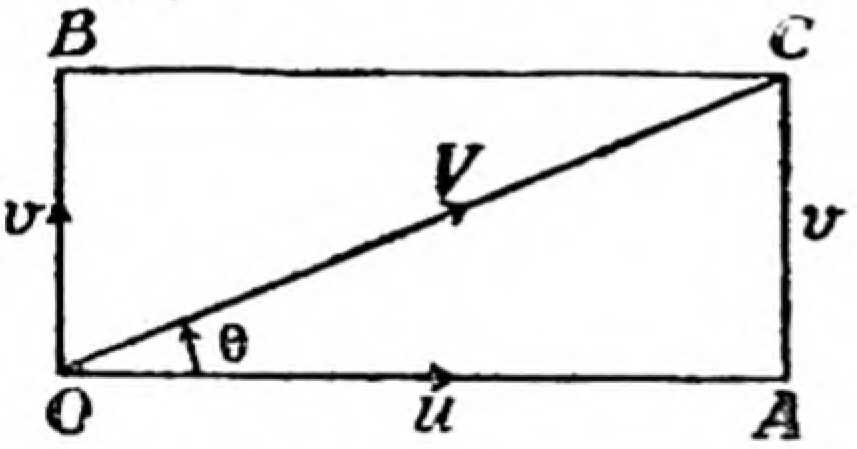
or $$V^2=u^2+v^2+2uv\cos\alpha$$

$\therefore$ $$V=\sqrt{u^2+v^2+2uv\cos\alpha}.$$

(*iii*) Let angle AOC be θ. Draw CL perpendicular to OA.

Then $$\tan\theta=\frac{CL}{OL}=\frac{CL}{OA+AL}=\frac{AC\sin\alpha}{OA+AC\cos\alpha}$$
$$=\frac{v\sin\alpha}{u+v\cos\alpha}.$$

Cor 1. If the components u and v are at right angles, $\alpha=90^\circ$, so that



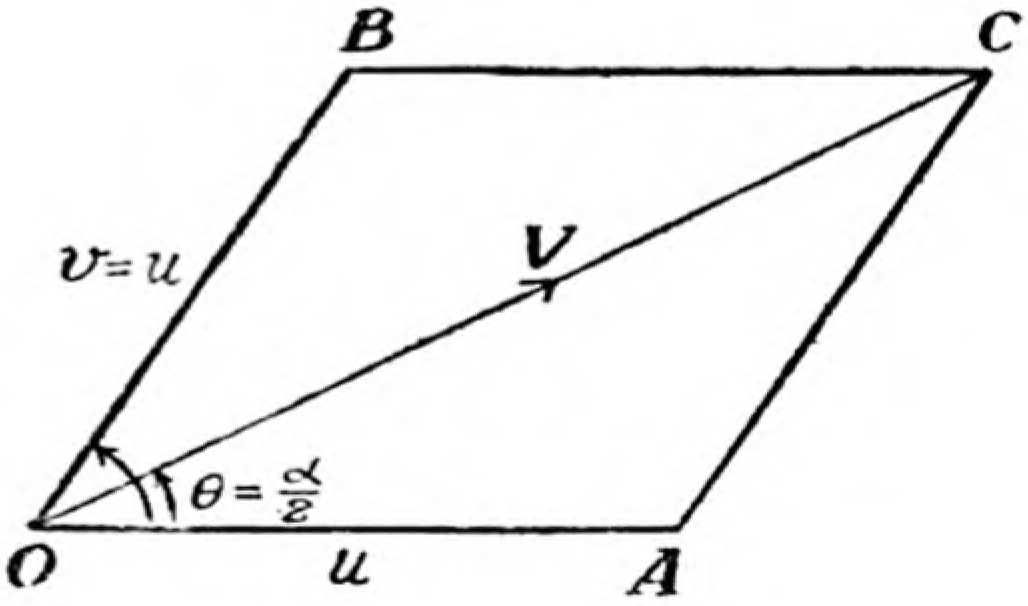
$$V^2=u^2+v^2$$

or $$V=\sqrt{u^2+v^2}$$

and $$\tan\theta=\frac{v}{u}.$$

Cor 2. If $v=u$,

$$V=\sqrt{u^2+u^2+2u^2\cos\alpha}$$
$$=u\sqrt{2+2\cos\alpha}$$
$$=2u\cos\frac{\alpha}{2}$$
$$\tan\theta=\frac{u\sin\alpha}{u+u\cos\alpha}$$
$$=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$$
$$=\tan\frac{\alpha}{2}.$$

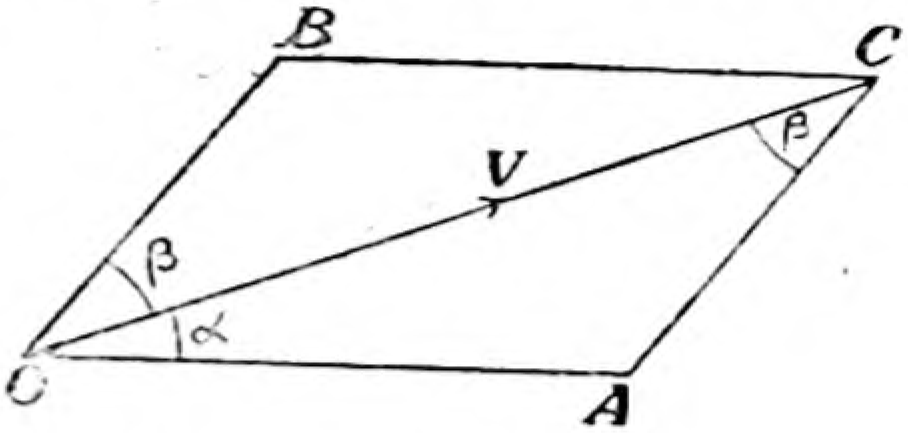


$\therefore$ $$\theta=\frac{\alpha}{2}.$$

Thus the line of action of the resultant of two velocities, equal in magnitude, bisects the angle between them.

3. **Resolution of Velocities** : Since an infinite number of parallelograms can be constructed with a given line as a diagonal, it follows that a given velocity can be resolved into two components in an infinite number of ways.

4. Components of a velocity V in the given directions making angles α, β with V : Let the velocity V be represented by OC. Draw OA and OB making angles α, β with OC. From C draw AC and BC parallels to OB and OA respectively. Then OA and OB represent the components of the velocity V.



In triangle OAC,

$$\frac{OA}{\sin\beta}=\frac{AC}{\sin\alpha}=\frac{OB}{\sin\alpha}=\frac{OC}{\sin(\alpha+\beta)}.$$

$$\therefore \qquad OA=\frac{V\sin\beta}{\sin(\alpha+\beta)}$$

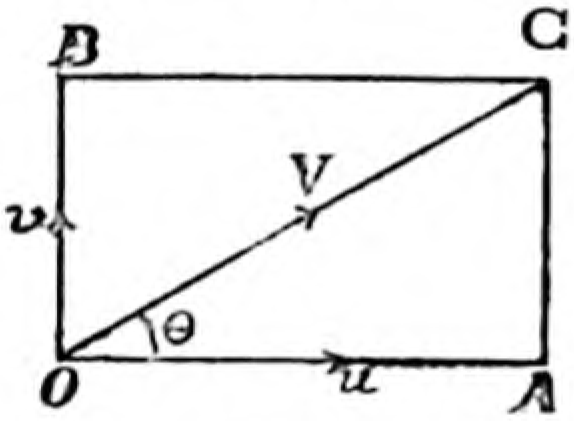
and

$$OB=\frac{V\sin\alpha}{\sin(\alpha+\beta)}.$$

5. Resolved Parts of a Velocity : The most important case arises when the velocity V is to be resolved into two velocities in two directions at right angles.

Let $\alpha=\theta.$

Then $\beta=90^\circ-\theta.$



Then the components make angles θ and $90^\circ-\theta$ with OC. In this case these components are called the **Resolved parts** of the velocity V. If these resolved parts are represented by u and v respectively,

$$u=V\cos\theta$$

and

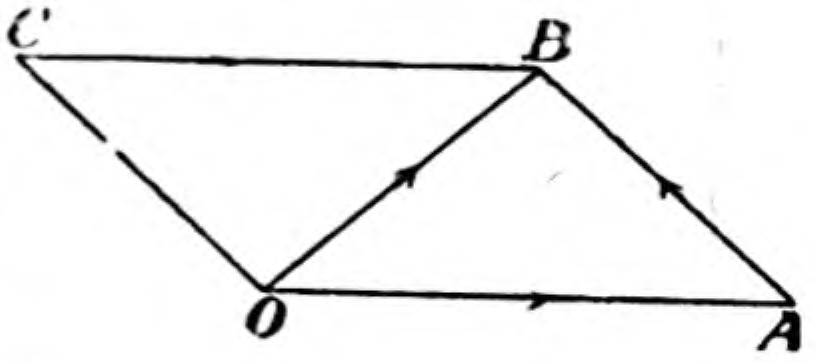
$$v=V\sin\theta=V\cos(90^\circ-\theta).$$

Thus the resolved parts of a given velocity in a certain direction is the product of the given velocity and the cosine of the angle between the direction of the velocity and the given direction.

Note. It should be clear that whereas the components of a velocity require two directions into which it is to be decomposed, only one direction is needed to find the resolved parts of the velocity, the other being at right angles to the first.

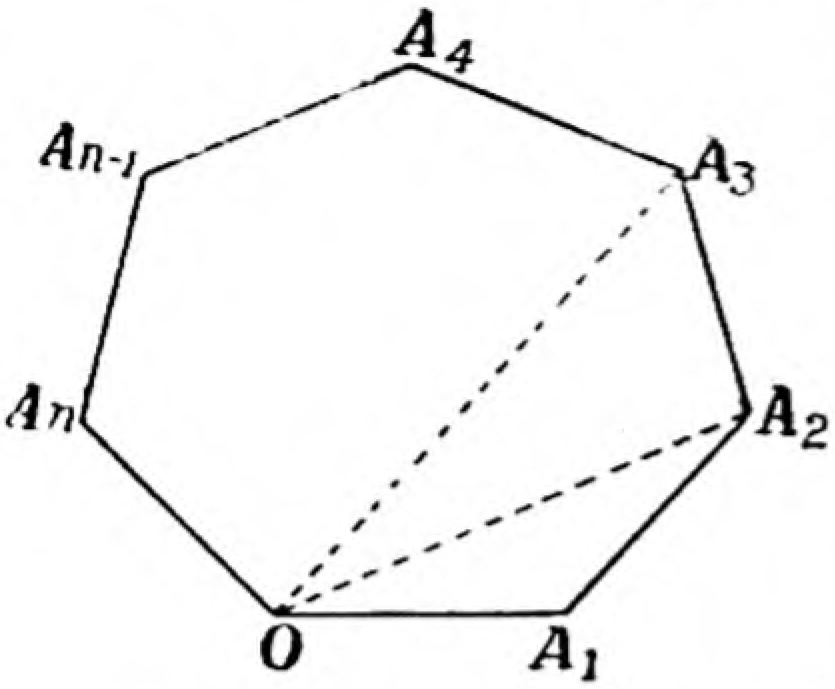
6. Law of Triangle of Velocities : If a moving point possesses simultaneously two velocities represented in magnitude and direction by the two sides of a triangle taken in order, their resultant is represented by the third side taken in the opposite order.

Let the given velocities be represented by the sides OA, AB of the triangle OAB. Complete the parallelogram OABC. Then OC being equal and parallel to AB represents the same velocity AB. Hence by the parallelogram law of velocities, the resultant is represented by OB.



Vectorially, we may write $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.

7. Law of Polygon of Velocities : If a point possesses simultaneously velocities represented by the sides OA_1, A_1A_2, A_2A_3,........., $A_{n-1}A_n$ of a polygon $OA_1A_2A_3$.........$A_{n-1}A_n$ taken in order, the resultant velocity is represented by OA_n.



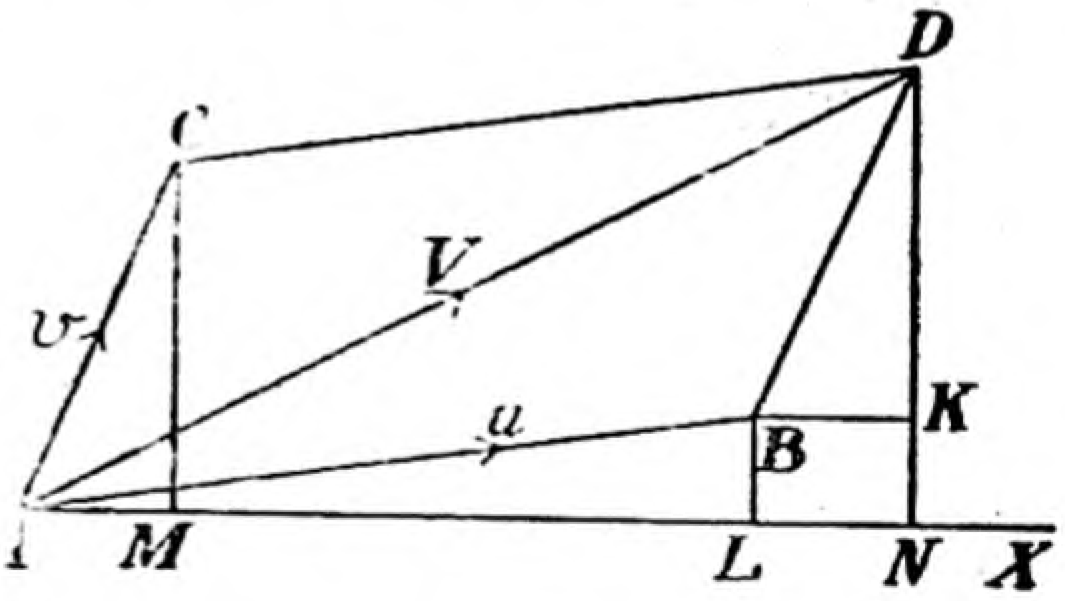
By the triangle of velocities, in Vector Notation :

$$\overrightarrow{OA_1} + \overrightarrow{A_1A_2} = \overrightarrow{OA_2}$$
$$\overrightarrow{OA_2} + \overrightarrow{A_2A_3} = \overrightarrow{OA_3}$$

and so on, till we get the final resultant.

8. *The sum of the resolved parts of the velocities in a given direction is equal to the resolved part of the resultant in the same direction.*

Let the velocities u and v be represented by AB and AC. Complete the parallelogram ABDC. Then the resultant V is represented by the diagonal AD.



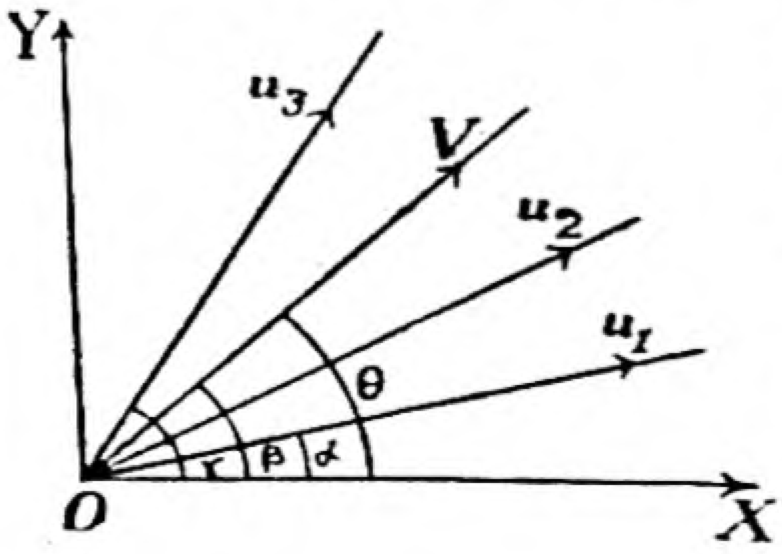
Through A draw a line AX parallel to the given direction. Draw BL, CM and DN perpendiculars to AX, and BK perpendicular to DN.

Now AL, AM and AN are, respectively, the resolved parts of u, v and V along OX.

$$\begin{aligned} AN &= AL + LN \\ &= AL + BK \\ &= AL + AM \end{aligned}$$

$\therefore$ resolved part of V along AX = resolved part of u along AX + resolved part of v along AX.

9. Resultant of a number of velocities of a particle : Let the velocities be u_1, u_2, u_3........... making angles α, β, γ ......with OX where OX and OY are two axes at angles. Let the resultant V make an angle θ with OX.



Then resolving along OX and OY, we get (by Art. 8)

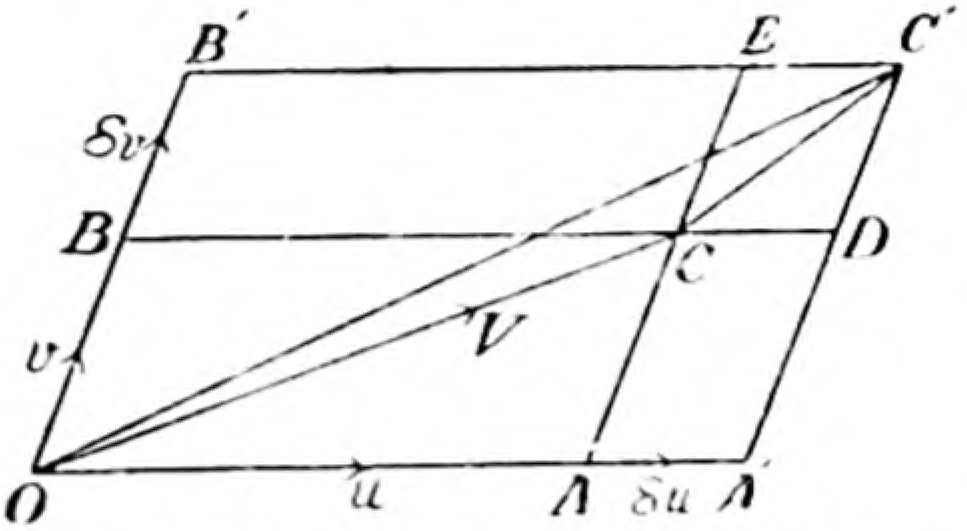
$$V \cos\theta = u_1 \cos\alpha + u_2 \cos\beta + \ldots = u \ldots \text{ (1)}$$
$$V \sin\theta = u_1 \sin\alpha + u_2 \sin\beta + \ldots = v \ldots \text{ (2)}$$

$$\therefore \qquad V=\sqrt{u^2+v^2}$$

and

$$\tan\theta=\frac{v}{u}\cdot$$

10. Law of Parallelogram of Accelerations : If a moving point possesses simultaneously two accelerations represented in magnitude and direction by the two sides of a parallelogram drawn from a point, then the resultant acceleration is represented in magnitude and direction by the diagonal of the parallelogram passing through the point.



Let at time t, the position of the point be O moving with two simultaneous velocities u and v represented in magnitude and direction by OA and OB respectively. Complete the parallelogram OACB. Let the changes in these velocities during the short interval δt be δu and δv. Let these be represented by AA′ and BB′ respectively. Then OA′ and OB′ represent the velocities at times $t+\delta t$. Complete the parallelogram OA′C′B′. Produce BC and AC so as to meet A′C′ and B′C′ at D and E respectively.

Then $\qquad CD=AA'=\delta u$ and $CE=BB'=\delta v$.

Now V, the resultant velocity at time t, is represented by OC and $V+\delta V$, the resultant velocity at time $t+\delta t$, is represented by OC′.

$\therefore$ by the parallelogram law of velocities, δV, the change in resultant velocity in time δt, is represented by CC′.

$\therefore$ if the changes δu, δv in the two velocities of a point be represented by the sides CD, CE respectively of a parallelogram CDC′E, then the change δV in the resultant velocity is represented by the diagonal CC′,

Now $\qquad \dfrac{\delta u}{\delta t}, \; \dfrac{\delta v}{\delta t}$ and $\dfrac{\delta V}{\delta t}$

are respectively proportional to δu, δv and δV.

$\therefore$ if $\dfrac{\delta u}{\delta t}, \dfrac{\delta v}{\delta t}$ are represented by the two sides of a parallelogram drawn through a point, $\dfrac{\delta V}{\delta t}$ can be represented by the diagonal of the parallelogram passing through that point. This is true however small δt may be.

$\therefore$ in the limit when $\delta t \to 0$, if the accelerations $\dfrac{du}{dt}$ and $\dfrac{dv}{dt}$ are represented in magnitude and direction by the two sides of a parallelogram passing through a point, then the resultant acceleration

$\frac{dV}{dt}$ is represented by the diagonal of the parallelogram passing through that point.

It should be noted that the accelerations are resolved and compounded in exactly the same way as velocities.

11. Relative Velocity: When the distance between two points is altering either in magnitude or direction or in both, each point is said to have a velocity relative to the other.

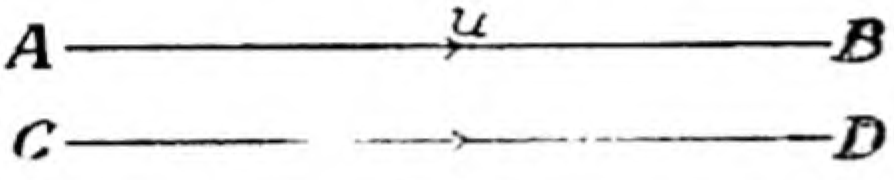
The relative velocity of A with respect to B is obtained by compounding the velocity of A with the reversed velocity of B.

Illustrations :

(*i*) If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the velocities of points P and Q. AB being parallel to CD.

$$AB = u, \qquad CD = v.$$

the velocity of P relative to Q is $u - v$, and that of Q relative to P is $v - u$.

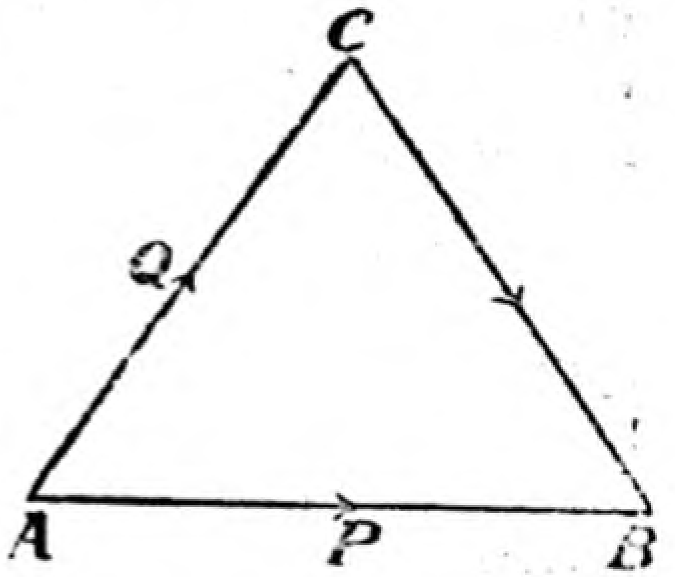


(*a*)

(*ii*) If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ represent the velocities of P and Q, [fig. (*b*)], the velocity of P relative to Q is

$$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$$

the velocity of Q relative to P $= \overrightarrow{BC}$. Hence we may obtain the velocity of P relative to Q by combining with the velocity of P a velocity equal and opposite to that of Q and the point Q may then be regarded as at rest.



(*b*)

Ex. A point has velocities 4, $3\sqrt{2}$, 2, and 1 inclined at angles of 30°, 45°, 60° and 120° respectively to a given direction. Find the magnitude and direction of their resultant.

Let OX be the given direction and OY perpendicular to it. The components along OX are

$$V\cos\theta = 4\cos 30° + 3\sqrt{2}\cos 45° + 2\cos 60° + 1\cos 120°$$

$$= 2\sqrt{3} + \frac{3\sqrt{2}}{\sqrt{2}} + 1 - \frac{1}{2} = \frac{7}{2} + 2\sqrt{3}.$$

$$V\sin\theta = 4\sin 30° + 3\sqrt{2}\sin 45° + 2\sin 60° + 1\sin 120°$$

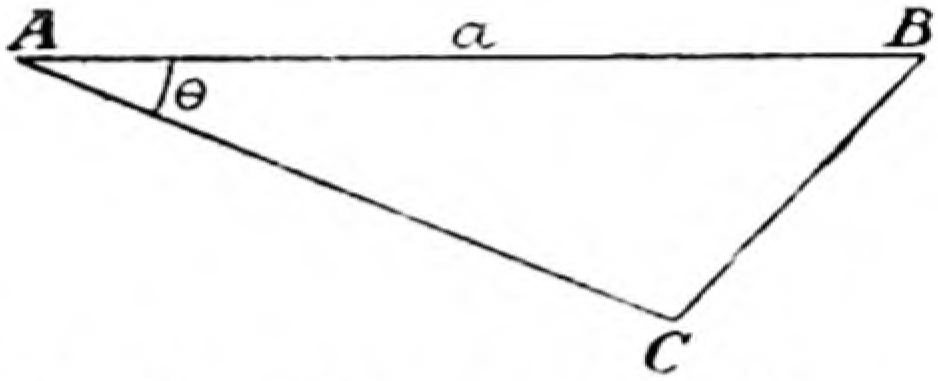
$$= 2 + 3 + \sqrt{3} + \frac{\sqrt{3}}{2} = 5 + \frac{3\sqrt{3}}{2}$$

$$V^2 = \left(\frac{7}{2} + 2\sqrt{3}\right)^2 + \left(5 + \frac{3\sqrt{3}}{2}\right)^2$$

$$\therefore \quad V = 10{\cdot}3$$

$$\tan\theta = \frac{10 + 3\sqrt{3}}{7 + 4\sqrt{3}}.$$

Ex. 5. If the distance between two moving points at any line be a, if V be their relative velocity and if u and v be the components of V respectively in and perpendicular to the direction of a, show that their distance when they are nearest to one another is $\frac{av}{V}$ and that the time that elapses before they arrive at their nearest distance is $\frac{au}{V^2}$.



Let A and B be the positions of the moving points at any time when AB is a.

Let AC be the path of A relative to B, *i.e.*, the path which A would appear to describe as observed from B when B is reduced to rest. The path AC is described by A with uniform velocity V.

The two points are nearest each other when A arrives at C the foot of the perpendicular from B. The distance between them then is $BC = a \sin\theta$.

But $V\cos\theta = u$ and $V\sin\theta = v$.

Hence $BC = \frac{av}{V}$.

The time for describing $AC = \frac{AC}{V}$

$$= \frac{a\cos\theta}{V} = \frac{au}{V^2}.$$

Exercise

1. A boat is rowed with a velocity of 6 miles per hour straight across a river which flows at the rate of 2 miles per hour. If the breadth is 300 feet, find how far down the river the boat will reach the opposite bank below the point at which it was originally directed.

2. A boat is rowed with a velocity of 4 m.p.h. straight across a river which is flowing at 3 m.p.h. Find the magnitude and direction of the resultant velocity of the boat. If the breadth of the river is 400 feet, find how far down the river the boat will reach the opposite bank.

3. Three velocities 4, 6 and 8 ft. per second act in directions making 30°, 135°, 210° with the horizontal. Find the resultant in magnitude and direction.

4. A point which has velocities represented by 8, 9 and 13 is at rest. Find the angle between the directions of the two smaller velocities.

5. If a point has two velocities u_1 and u_2 inclined at such an angle that the resultant velocity $V = u_1$, show that if u_1 be doubled, the new resultant is at right angles to u_2.

6. A ship is steaming due east at 15 m.p.h. and another ship is steaming due south at 20 m.p.h. Find the velocity of the second ship relative to the first.

7. Two trains are travelling on lines which cross at right angles, one at 40, and the other at 50 m.p.h. Find the velocity of the second train relative to the first.

8. A train is travelling along a horizontal rail at 45 m.p.h. and rain is falling vertically with a velocity of 22 ft./sec. Find the apparent direction and velocity of the rain to a person travelling in the train.

ANSWERS

Exercise

1. 100 ft.
2. 5 m./h. at $\tan^{-1} \frac{4}{3}$ with the direction of flow of the river, 300 ft.
4. 80° 25′.
6. 25 m./h. and makes an angle $\tan^{-1} \frac{3}{4}$ West of South.
7. 64·03 m./h. at 38° 40′ with direction of second.
8. $\tan^{-1} 3$ with direction of actual velocity of rain, $22\sqrt{10}$ ft./sec.